THE

SYSTEM OF THE WORLD,

BY

M. Le MARQUIS DE LAPLACE,

TRANSLATED FROM THE FRENCH,

AND

ELUCIDATED WITH EXPLANATORY NOTES.

BY THE

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CONTENTS.

BOOK I.

OF THE APPARENT MOTION OF THE HEAVENLY BODIES.

BOOK II.

OF THE REAL MOTIONS OF THE HEAVENLY BODIES.

BOOK III.

OF THE LAWS OF MOTION.

PREFACE

It has been made a matter of surprise, that notwithstanding there are many individuals in these countries perfectly competent to the task, there has not as yet appeared a translation of the works of Laplace.

That an accurate translation of the works of this great man would render them more easily apprehended, and would also contribute to their being more extensively known, cannot be questioned by any person who considers, that they are read with avidity by many persons who are frequently embarrassed as to the author's meaning, in consequence of their imperfect acquaintance with the French language. The present Translation was drawn up for the purpose of obviating these difficulties, and of rendering the work accessible to every scientific student. It is

hoped that the Notes which are subjoined at the end of each volume will tend to elucidate many of the important results which are merely announced in the text. The Translator is aware, that to those readers who are conversant with the Celestial Mechanics, many, if not all, of these might be dispensed with; but when it is considered, that his object has been to render these objects accessible to the generality of readers, he trusts he will not be deemed unnecessarily diffuse, if he has insisted longer on some points than the experienced reader would think necessary.

The decimal division of the circle, and of the day, (of which the origin is fixed at midnight,) is adopted in the text. The lineal measures are referred to the metre, and all temperatures are estimated on the centigrade thermometer, the height of the barometer being supposed equal to 76 centimetres, when this thermometer points to zero at the parallel of 45°.

By means of the following table, any number of decimal degrees, minutes, and seconds, may be obtained in sexagesimal degrees, minutes, and seconds, by simple multiplication :

Decimal.	Sexagesimal.		Sexages.	Decimal.
1° = 54′	= 324″		1° = 1° 11′ 11″ 11‴, &c.	
1′ = 32″,4			1′ = 1′, 85″ 18‴. 51, &c.	
1″ = 0″,324			1″ = 3″ 8‴. 64.	

As it is frequently required to know the values of the corresponding quantities, according to the English standard of weights, measures, &c., the following table is subjoined, by means of which it is extremely easy to estimate the French measures in terms of the English, or *vice versa*.

1 foot = 12 inches,	= 12.785 inches.
∴ 3 feet + $1\frac{3}{7}$ of an inch,	= 3 feet, or one yard, which is the English linear standard.
The *metre* = 10,000,000 of the distance of the pole from the equator,	= 39.383 inches.
The *litre*, which is the unit of capacity, (= the cube of the tenth part of the metre,)	= 61.083 inches.
The *gramme*, which is the unit of weight, (= the weight of a cube of distilled water, of which one side is the 100th part of the metre,	= 22.966 grains.
The *are*, which is the superficial measure,	= 11. 968 square yards.

The following numerical values being of frequent occurrence will likewise be useful to the practical student: l denoting the logarithm of a quantity in the Hyperbolic or Naperian system, of which the modulus = 1, and L denoting the logarithm of a quantity in the common system, of which the base = 10, we have e, the base of

the Hyperbolic system $= 2,\ 7128\ 18284\ 59045\ 23536$, &c., the modulus in the common system $= Lc = 0,\ 43429,\ 44819\ 03251\ 827651\ 11289$.

The ratio of the diameter to the periphery of a circle, or π the semiperiphery of a circle, of which the rad. is unity =

$$
\begin{array}{rl}
 & 3,\ 14159\ 26535\ 89793\ 23846\ 26433\ 83279 \\
L.\ \pi = & 0,\ 49714\ 98726\ 94133\ 85635\ 127 \\
l.\ \pi = & 1\ \ 14472\ 98858\ 49400\ 17414\ 342
\end{array}
$$

In our division of the day, one second of time is the 86400th part of the mean day. In the present French division, one second is the 100,000th part of the mean day, $\therefore$ denoting by g the force of gravity, and by λ the length of the pendulum which vibrates seconds. In the latitude of Paris we have

$$
\left.\begin{array}{rl}
g. = & 9^{m},\ 808795248 \\
L.\ g. \ \ & 0,\ \ 9916156690 \\
\lambda. = & 0^{m},\ 993838746 \\
L.\ \lambda. = & 1,\ \ 9973159236
\end{array}\right\}\text{In the ancient division of time.}
\quad
\left\{\begin{array}{l}
= 7^{m},\ 32214 \\
= 2\ \ \ 8646381 \\
= 0^{m},\ 741887 \\
= 1,\ \ 8703378
\end{array}\right\}\text{In the present French division of time.}
$$

THE

SYSTEM OF THE WORLD.

Me vero primum dulces ante omnia musæ
Quarum sacra fero, ingenti perculsus amore,
Accipiant, cælique vias et sidera monstrent.
VIRG. *lib.* II, GEOR.

OF all the natural sciences, astronomy is that which presents the longest series of discoveries. The first appearance of the heavens is indeed far removed from that enlarged view, by which we comprehend at the present day, the past and future states of the system of the world. To arrive at this, it was necessary to observe the heavenly bodies during a long succession of ages, to recognize in their appearances the real motion of the earth, to develope the laws of the planetary motions, to derive from these laws the principle of universal gravitation, and finally from this principle to descend to the complete explanation of all the celestial phenomena in their minutest details. This

is what the human understanding has atchieved in astronomy. The exposition of these discoveries, and of the most simple manner, in which they may arise one from the other, will have the twofold advantage of furnishing a great assemblage of important truths, and of pointing out the true method which should be followed in investigating the laws of nature. This is the object which I propose in the following work.

BOOK THE FIRST.

OF THE APPARENT MOTIONS OF THE HEAVENLY BODIES.

CHAP. I.

Of the diurnal motion of the heavens.

IF during a fine night, and in a place where the view of the horizon is uninterrupted, the appearance of the heavens be attentively observed, it will be perceived to change at every instant. The stars are either rising above or descending towards the horizon; some appear towards the east, others disappear towards the west; several, as the pole star, and the stars of the great Bear, never reach the horizon in our climates. In these various motions, the relative position of all the stars remains the same: they describe circles which diminish in proportion as they are nearer to a point which seems to be immoveable. Thus the heavens appear to revolve about two fixed points, termed from this circumstance, *poles of the world*; and in this motion they are supposed to carry with them, the entire system of the stars. The pole which is elevated above the horizon is the *north*

pole. The opposite pole, which we imagine to be depressed beneath the horizon, is the *south* pole.

Already several interesting questions present themselves to be resolved. What becomes during the day of the stars which have been seen in the night? From whence do those come which begin to appear? and where are those gone which have disappeared? An attentive examination of the phenomena furnishes very simple answers to these questions. In the morning the light of the stars grows fainter, according as the dawn advances; in the evening they become more brilliant, as the twilight diminishes; it is not therefore because they cease to shine, but because they are effaced by the more vivid light of the twilight and of the sun, that we cease to perceive them.

The fortunate discovery of the telescope has furnished us with the means of verifying this explanation, for the stars seen through this instrument are visible, even when the sun is at its greatest elevation above the horizon. Those stars, which from their proximity to the pole, never reach the horizon, are perpetually visible. With respect to the stars which rise in the east and set in the west, it is natural to suppose that they complete under the horizon the circle, part of which appeared to be described above it. This truth become more obvious as we advance towards the north, more and more of the stars situated in this part of the world are extricated from beneath the horizon, till at length these stars cease to disappear at all, while the stars

which are situated towards the south become entirely invisible. When we advance towards the south pole, the contrary is observed to be the case; stars which always continued above the horizon, commence to rise and set alternately, and new stars previously invisible begin to appear. It appears from these phenomena that the surface of the earth is not what it appears to be, namely, a plane on which the celestial vault is supported. This is an illusion which the first observers rectified very soon, by considerations similar to the preceding; they observed that the heavens surround the earth on all sides, and that the stars shine perpetually, describing every day their respective circles. We shall have frequent occasion to observe in the sequel, cases in which similar illusions have been dissipated, and in which even the real objects have been recognized in their erroneous appearances, by means of astronomy.

In order to form an accurate conception of the motion of the stars, we conceive an axis to pass through the centre of the earth, and the two poles of the world, on which the celestial sphere revolves. The great circle perpendicular to this axis is called the Equator, the lesser circles which the stars describe parallel to the equator, in consequence of their diurnal motion, are termed parallel circles. The *zenith* of a spectator, is that point of the heavens to which his vertical is directed. The *nadir* is the point diametrically opposite. The *meridian* (*a*) is the great circle

which passes through the *zenith* and the *poles*; it divides into two equal parts the arcs described by the stars above the horizon, so that when they are on this circle, they are at their greatest or least altitude. Finally, the *horizon* is the great circle perpendicular to the vertical, or parallel to the surface of stagnant water at the place of the observer.

The elevation of the pole being an arithmetic mean between the greatest and least altitudes of the stars which never set, an easy method is suggested of determining the height of the pole. As we advance directly towards the pole, it is observed to be elevated very nearly in proportion *(b)* to the space passed over; hence it is inferred that the surface of the earth is convex, its figure differing little from that of a sphere. The curvature of the terrestrial globe is very sensible on the surface of the seas; the sailor in his approach towards the shore perceives first the most elevated points, and afterwards the lower parts, which were concealed from his view by the convexity of the earth. It is also in consequence of this curvature, that the sun at its rising gilds the summits of the mountains before he illuminates the planes.

CHAP. II.

Of the sun, and of its motions.

All the heavenly bodies participate in the diurnal motion of the celestial sphere, but several have proper motions of their own, which it is interesting to follow,* because it is by means of these alone, that we can hope to arrive at the knowledge of the true system of the world. As in measuring the distance of an object, we observe it from two different positions, so in order to discover the mechanism of nature, we must consider her under different points of view, and observe the development of her laws, in the changes of appearance which she presents to us. Upon the earth, we vary the phenomena by experiments, in the heavens we carefully determine all those which the celestial motions present to us. By thus interrogating nature, and subjecting her answers to analysis, we can by a train of inductions judiciously managed, arrive at the general phenomena, from whence these particular facts arise. It is to discover these grand phenomena, and to reduce them to the least possible number, that all our efforts should be directed; for the first causes and intimate nature of beings will be for ever unknown.

The sun has a proper motion, of which the direction is contrary to the diurnal motion. This motion is recognised by the appearances which the heavens present during the nights, which appearances change and are renewed with the seasons. The stars situated in the path of the sun, and which set a little after him, are very soon lost in his light, and at length reappear before his rising; this star therefore advances towards them, from west to east. It is thus that for a long time his proper motion was traced, (which at present can be determined with great precision), by observing every day, the meridian altitude of the sun, and the interval of time which elapses between his passage, and that of the stars over the meridian. By means of these observations, we obtain the proper motions of the sun, in the direction of the meridian, and also in the direction of the parallels; the resultant of these motions is the true motion of this star about the earth. In this manner, it has been found that this star moves in an orbit, which is called the *ecliptic*, and which at the commencement of 1801, was inclined to the equator at an angle of 26°,07315.

The variety of seasons is caused by the inclination of the ecliptic to the equator. When the sun in his annual motion arrives at the equator, he describes very nearly in his diurnal motion this great circle, which being then divided into two equal parts by all the horizons, the day is equal to the night, in every part of the earth. The points of the intersection of the equator and

the ecliptic, are termed *the equinoxes*, on account of this equality. In proportion as the sun, after leaving the equinox of spring, advances in his orbit, his meridian altitudes above our horizon increase, the visible arc of the parallels, which it describes every day, continually increases, and this augments the length of the days, till the sun has attained his greatest altitude. At this epoch, the days are the longest in the year, and because the variations of the meridian height of the sun, are insensible, near the *maximum*, the sun (considering only the altitude on which the duration of the day depends) appears stationary, for which reason, (*c*) this point of the *maximum* height has been termed the summer solstice. The parallel described by the sun on that day, is called the summer *tropic*. This star then descends towards the equator, which it traverses again, at the autumnal equinox, from thence it arrives at its *minimum* of altitude, or at the winter *solstice*. The parallel then described by the sun is the winter *tropic*, and the corresponding day is the shortest of the year; having attained this term, the sun again ascends and returns to the vernal equinox, to recommence the same route.

Such is the constant regular progress of the sun and of the seasons. Spring, is the interval comprised between the vernal equinox, and the summer solstice; summer is the interval from this solstice to the autumnal equinox; and the interval from the autumnal equinox to the winter solstice, constitutes the autumn; finally, winter is

the interval of time from the winter solstice to the vernal equinox.

The presence of the sun above the horizon being the cause of heat, it might be supposed that the temperature should be the same in summer as in spring, and in the winter and autumn. But the temperature is not the instantaneous effect of the presence of the sun, it is rather the result of its long continued action. It does not produce its *maximum* of effect, for each day, till some time after the greatest altitude of this star above the horizon, nor does it attain its maximum effect for the year, till the greatest solstitial altitude is passed.

The different climates exhibit remarkable varieties, which we will now examine from the equator to the poles. At the equator, the horizon divides all the parallels into two equal parts; the day is therefore constantly equal to the night. In the equinoxes the sun, at mid day, passes through the zenith. The meridian altitudes of this star, at the solstices, are least, and equal to the complement of the inclination of the ecliptic to the equator. The solar shadows are then directly opposite, which is never the case in our climates, where they are always at mid-day directed towards the north.

At the equator, therefore, properly speaking, there are two summers and two winters, every year. This is also the case in all places, where the height of the pole is less than the obliquity of the ecliptic. Beyond this limit, as the sun never can be in the zenith, there is only one summer and one winter in

each year; the duration of the longest day increases and that of the shortest day diminishes as we approach the pole, and at the parallel the distance of the zenith of which from the pole, is equal to the obliquity of the ecliptic, the sun never (*d*) sets on the day of the summer solstice, nor rises on the day of the winter solstice. Still nearer to the pole, the time of his presence, and of its absence, exceeds several days, and even months. Finally, under the pole, the horizon coinciding with the equator itself, the sun is always above the horizon when on the same side of the equator as the pole; it is constantly below the horizon, when it is at the other side of the equator; so that there is then but one day and one night throughout the year. (*e*)

Let us trace more particularly the path of the sun. It is at once apparent that the intervals which separate the equinoxes and the solstices are unequal, that from the vernal to the autumnal equinox, is about eight days longer than the interval between the autumnal and vernal equinoxes; the motion is consequently not uniform: by means of accurate and repeated observations, it has been ascertained that the motion is most rapid in a point of the solar orbit, which is situated near the winter solstice, and that it is slowest in the opposite point of the orbit near to the summer solstice. The sun describes in a day 1°,1327 in the first point, and only 1°,0591 in the second. Thus during the course of the year its motion varies from the greatest to

the least by three hundred and thirty-six ten thousandths of its mean value. (*f*)

This variation produces, by its accumulation, a very sensible inequality in the motion of the sun. In order to determine its law, and in general to obtain the laws of all the periodical inequalities, it should be remarked that these inequalities may be properly represented by the sines and cosines of angles which become the same after the completion of every circumference. (*g*) If therefore all the inequalities of the celestial motions are expressed in this manner, the only difficulty consists in separating them from each other, and in determining the angles on which they depend. As the inequality which we are at present considering, performs the period of its changes in a revolution of the sun, it is natural to make it depend on the motion of the sun and on its multiples. In this manner, it has been found that it is expressed by means of a series of sines depending on this motion; it is reduced very nearly to two terms, of which the first is proportional to the sine of the mean angular distance of the sun, from the point in its orbit, where his velocity is greatest, and of which the second is about ninety five times less than the first, and proportional to the sine of double of this distance.

It is probable that the distance of the sun from the earth varies with its angular velocity, and this has been proved by the measures of its apparent diameter. This diameter increases and diminishes

according to the same law as the velocity, but in a ratio only half as great. When the velocity is greatest, the diameter is 6035,″8 and it is observed to be only 5836,″3, when this velocity is the least; therefore its mean magnitude is about 5936,″0.

The distance of the sun from the earth being reciprocally proportional to his apparent diameter, its increase follows the same law as the diminution of this diameter. The point of the orbit in which the sun is nearest to the earth, is termed the *perigee*, and the opposite point, in which the sun is most remote, is called the *apogee*. It is in the first of these points, that the apparent diameter and also the velocity of the sun are greatest; in the second point, the apparent diameter and velocity are at their *minimum*.

It would be sufficient, in order to explain the diminution of the sun's apparent motion, to increase his distance from the earth; but if the variation of the solar motion arose from this cause only, and if the real velocity of the sun was constant, its apparent velocity would diminish in the same ratio as the apparent diameter. It diminishes in a ratio twice as great, therefore there is an actual retardation in the motion of this star, when it recedes from the earth. From the effect of this retardation, combined with the increase of distance, its angular motion diminishes as the square of the distance increases, so that its product by this square is very nearly constant. All the measures of the apparent diameter of the sun, compared with the observations of his daily motion, confirm this result.

Let us conceive a right line joining the centres of the sun and earth, which we will call the *radius vector* of the sun, it is easy to perceive that the small sector, or area described in a day by this radius about the earth, is proportional to the square of this radius into the diurnal motion (h) of the sun. This area is therefore constant, and the entire area described by the radius vector, reckoning from a given point, increases as the number of days, elapsed since the epoch at which the sun was on this radius. *Therefore the areas described by its radius vector, are proportional to the times.* This simple relation between the motion of the sun, and its distance from the focus of this motion, must be admitted as a fundamental law in its theory, at least, until observations compel us to modify it.

If from the preceding data, the position and length of the radius vector of the solar orbit be set down every day, and a curve be supposed to pass through the extremities of all those radii, it will appear that this curve will be somewhat elongated in the direction of the right line, which, passing through the centre of the earth, joins the points of the greatest and least distance of the sun. The resemblance of this curve with the ellipse, having suggested the notion of comparing them together, their identity was ascertained; from which it has been inferred, *that the solar orbit is an ellipse, of which the centre of the earth occupies one of the foci.* (i)

The ellipse is one of those curves so celebrated both in antient and modern geometry, which being formed by the intersection of a plane with the surface of a cone, have been therefore termed *conic sections.* The extremities of a thread which is stretched on a plane, being fixed on two immovable points, called foci, any point which slides along this thread describes the ellipse; it is evidently elongated in the direction of the right line which joins the foci, and which being extended on each side to meet the curve, forms the greater axis, of which the length is equal to that of the thread. The lesser axis is the right line drawn through the centre perpendicularly to the greater axis, and extended on both sides to meet the curve: the distance of the centre from one of the *foci* is the *excentricity* of the ellipse. When the two foci are united in the same point, the ellipse becomes a circle; by increasing their distance the ellipse gradually lengthens, and if the mutual distance becomes infinite, the distance of the focus from the nearest summit of the curve, remains finite, and the ellipse becomes a parabola.

The solar ellipse differs but little from a circle; for the excess of the greatest above the least distance of the sun from the earth is equal to the hundred and sixty ten thousandth part of this distance. This excess is the excentricity itself, in which observations indicate a very slow diminution, and hardly perceptible in a century.

In order to have a just conception of the elliptic

motion of the sun, let us conceive a point to move uniformly on the circumference of a circle, of which the centre coincides with the centre of the earth, the radius being equal to the perigeon distance of the sun; suppose moreover that this point and the sun set off together from the perigee, and that the angular motion of the point is equal to the mean angular motion of the sun, while the radius vector of this point revolves uniformly about the earth, the radius vector of the sun moves unequally, always constituting with the distance of the perigee, and the arcs of the ellipse, sectors proportional to the times. At first, it precedes the radius vector of the point, and makes with it an angle, which after having increased to a certain limit (k), diminishes, and at length vanishes, when the sun arrives at his apogee. The two radii will then coincide with the greater axis. In the second half of the ellipse, the radius vector of the point precedes in its turn, that of the sun, and makes with it angles exactly equal to those, which it made in the first half, at the same distance from the perigee, at which point it coincides again with the radius vector of the sun, and with the greater axis of the ellipse. The angle by which the radius vector of the sun precedes that of the point, is termed *the equation of the centre*. Its *maximum* was 2°,13807 at the commencement of the present century, *i. e.* at the midnight, on which the first of January 1801 commenced. It diminishes by a quantity equal to about 53″ for every century.

From the duration of the sun's revolution in its orbit, the angular motion of the point about the earth may be inferred. The angular motion of the sun will be obtained by adding to this motion, the equation of the centre. The investigation of this equation is a very interesting problem of analysis, which can only be resolved by approximation; but the small excentricity of the solar orbit leads to very converging series, which are easily reduced to the form of tables.

The greater axis of the solar ellipse is not fixed in the heavens; it has relatively to the fixed stars an annual motion of about 36,″ in the same direction as that of the sun.

The solar orbit approaches by insensible degrees to the equator; the secular diminution of its obliquity, to the plane of this great circle, may be estimated at about 148″.

The elliptic motion of the sun does not exactly represent modern observations; their great precision has enabled us to perceive small inequalities, of which it would have been impossible to have developed the laws by observations alone. The investigation of these inequalities appertains to that branch of astronomy, which redescends from causes to the phenomena, and which will constitute the subject of the fourth book.

The distance of the sun from the earth, has at every period interested astronomers. Observers have (*l*) endeavoured to determine it, by all the means astronomy has successively furnished them with. The most natural and simple is that which

Geometers employ in measuring the distance of terrestrial objects. At the two extremities of a known base, the angles, which the visual rays of an object make with it, are observed, and by deducting their sum from two right angles, the angle will be obtained which these rays form at the point where they meet; this angle is termed the *parallax* of the object, the distance of which from the extremities of the base is easily obtained. In applying this method to the sun, the most extensive base which can be taken on the surface of the earth should be selected. Suppose two observers situated under the same meredian, and observing at noon, the distance of the centre of of the sun from the north pole; the difference of these two observed distances will be the angle, which the line joining the observers would subtend at this centre; the differences of the elevations of the pole gives this line in parts of the terrestrial radius; it will therefore be easy to infer from thence the angle under which the semidiameter of the earth would appear at the centre of the sun. This angle is the *horizontal parallax* of the sun; but it is too small to be accurately determined by this method, which only enables us to judge that the distance of this star is at least nine thousand diameters of the earth. In the sequel, it will be seen, that the discoveries in astronomy furnish other methods much more accurate for determining the parallax, which we now know to be about 27″, very nearly, at its

mean distance from the earth; hence it follows that this distance is about 23984 terrestrial radii.

Black spots are observed on the surface of the sun, of an irregular and variable form. Sometimes they are very numerous, and of considerable extent; some have been observed, of which the magnitude was equal to four or five times that of the earth. At other times, though rarely, the surface of the sun has appeared pure, and without spots for several successive years. Frequently the solar spots are enveloped by penumbras, which are themselves surrounded by a more brilliant light than that of the rest of the sun, in the middle of which these spots are observed to form and to disappear. The nature of these spots is yet unknown, however they have made us acquainted with a remarkable phenomenon, namely, the rotation of the sun. Amidst all the variations which they undergo in their position and magnitude, we can discover regular motions precisely the same as those of corresponding points of the surface of the sun, if we suppose it to have a motion of rotation in the direction of its motion round the earth, on an axis almost perpendicular to the ecliptic. From a continued observation of these spots, it has been inferred that the duration of an entire revolution of the sun is about twenty-five days and a half, and that the solar equator is inclined at an angle of eight degrees and one third to the plane of the ecliptic.

The extensive spots of the sun are almost al-

ways comprised in a zone of its surface, the breadth of which, measured on the solar meridian, does not extend beyond thirty-four degrees on each side of the equator; however, spots have been observed which were forty-four degrees from this equator. There has been observed, particularly about the vernal equinox, a faint light which is visible before the rising and after the setting of the sun, to which has been given the name of *zodiacal light.* Its colour is white, and its apparent figure that of a spindle, the base of which rests on the solar equator; such as would be the appearance of an ellipsoid of revolution extremely flattened, the centre and plane of equator coinciding with those of the sun. The length of this zodiacal light appears sometimes to subtend an angle of more than one hundred degrees. The fluid which reflects this light to us, must be extremely rare, since the stars are sometimes visible through it. The most received opinion respecting its nature is, that this fluid is the atmosphere itself of the sun; but this atmosphere certainly does not extend to so great a distance.—At the conclusion of this work we will suggest what appears to us to be the cause of this light, which is unknown, and has hitherto baffled our enquiries.

CHAP. III.

Of Time, and of its measure.

TIME is, relatively to us, the impression which a series of events, of which we are assured that the existence has been successive, leaves in the memory. Motion is a proper measure of time; for since a body cannot be in several places at the same time, when it moves from one place to another, it must pass successively through all the intermediate points. If it is actuated by the same force at every point of the line, which it describes, its motion is uniform, and the several portions of this line will measure the time employed to describe them. When a pendulum, at the termination of each oscillation, is in precisely the same circumstances as at the commencement of the motion, the durations of these oscillations are the same, and the time may be measured by their number. We may also employ for this measurement, the revolutions of the celestial sphere, in which the motions appear to be perfectly uniform; and mankind have universally agreed to make use of the motion of the sun for this purpose, the returns of which to the meridian, and to the same equinox or the same solstice, constitute the day and the year.

In civil life, the day is the interval of time which lapses from the rising to the setting of the sun: the night is the time, during which the sun remains below the horizon. The astronomical day comprises the entire duration of the diurnal revolution; it is the interval of time between two successive noons or midnights. It is greater than the duration of a revolution of the heavens, which constitutes the *sidereal day*; for if the sun and a star pass the meridian at the same instant, on the following day the sun will pass later, in consequence of its proper motion, by which it advances from west to east, and in the interval of a year it will pass the meridian once less than the star. It is found by assuming the mean astronomical day equal to unity, that the sidereal day is 0,99726957.

The astronomical days are not equal; their difference arises from two causes, namely, the inequality of the proper motion of the sun, and the obliquity of the ecliptic. The effect of the first cause is evident; thus, at the summer solstice, near to which the motion of the sun is slowest, the astronomical day approaches more to the sidereal day than at the winter solstice, when the motion is most rapid.

In order to conceive the effect of the second cause, it should be observed that the excess of the astronomical over the sidereal day arises solely from the proper motion of the sun reduced to the equator. If we conceive two great circles to pass through the poles of the world, and through the

extremities of the small arc which the sun describes on the ecliptic each day, the arc of the equator, which they intercept, is the daily motion of the sun referred to the equator, and the time which this arc takes to pass over the meridian, is the excess of the astronomical over the sidereal day; but it is evident that in the equinoxes, the arc of the equator is less than the corresponding arc of the ecliptic, in the ratio of the cosine of the obliquity of the ecliptic to radius; in the solstices it is greater in the ratio of radius to the cosine (m) of the same obliquity; therefore the astronomical day is diminished in the first case, and increased in the second.

To obtain a mean day, independent of these causes; we imagine a second sun, which moving uniformly in the ecliptic, passes always at the same instant as the true sun the greater axis of the solar orbit; this will cause the inequality of the proper motion of the sun to disappear. The effect arising from the obliquity is then made to disappear, by imagining a third sun to pass through the equinoxes at the same moment as the second sun, and to move on the equator in such a manner, that the angular distances of these two suns from the vernal equinox, may be constantly equal to each other. The interval of time between two consecutive returns of this third sun to the meridian, consitutes the mean astronomical day. *Mean time* is measured by the number of these returns, and the *true time* is

measured by the number of returns of the true sun to the meridian. The arc of the equator, intercepted between two meridians drawn through the centres of the true sun, and of the third sun, converted into time, in the proportion of the entire circumference to one day, is what is termed the (n) *equation of time.*

The day has been divided into twenty-four hours, and its origin has been fixed at midnight. The hour is divided into sixty minutes, the minutes into sixty seconds, the second into sixty thirds, &c. But the division of the day into ten hours, of the hours into one hundred minutes, of the minutes into one hundred seconds, will be adopted in this work, as being much more convenient for astronomical purposes.

The second sun, which we have imagined, determines by its returns to the equinoxes and the solstices, the mean equinoxes and solstices. The duration of its returns to the same equinox, or the same solstice, forms the *tropical year*, of which the actual length is about $365^{d},24226419$. Observation shews us that the sun employs a longer time to return to the same fixed stars. The *sidereal year* is the interval between two of these consecutive returns; it exceeds a tropical year by about $0^{d},014119$. Therefore the equinoxes have a retrograde motion on the ecliptic, or contrary to the proper motion of the sun, in consequence of which they describe every year, an arc equal to the mean motion of this star, in the interval of about $0^{d},014119$, which is very nearly

equal to 154″,63. This motion is not exactly the same every century, on which account, the duration of the tropical year is subject to a small inequality; it is now about 13″ shorter than in the time of Hipparchus.

It is natural that the year should be made to commence at one of the equinoxes or solstices; but if the origin of the year was placed at the summer solstice, or at the autumnal equinox, the same operations and labours would be appropriated to two different years. A like inconvenience would arise if the day was supposed to commence at noon, according to the custom of the old astronomers. It seems therefore most natural, that the year should be made to commence at the vernal equinox, at which period nature begins to revive; but it is equally natural to fix its commencement at the winter solstice, when, according to the received opinion of all antiquity, the sun begins to revive, and which is the middle of the longest night in the year under the poles.

If the length of the civil year was constantly 365 days, its commencement would always (*o*) anticipate that of the true tropical year, and it would pass through the different seasons with a retrograde motion in a period of about 1508 years. But this year (which was formerly in use in Egypt) would deprive the calendar of the advantage of attaching the months and festivals to the same seasons, and of rendering them useful epochs for the purposes of agriculture. This inestimable advantage would be secured, by con-

sidering the origin of the year as an astronomical phenomenon, which should be fixed by computation to the midnight which immediately precedes the equinox or the solstice: this has been done in France at the end of the last century. But then the bissextile years being intercalated according to a very complicated law, it would be difficult to resolve any given number of years into days, which would cause great confusion in history and chronology. Besides the origin of the year, which is always required to be known in advance, would be uncertain and arbitrary when it approached midnight, by a quantity less than the error (p) of the solar tables. Finally, the order of the bissextiles would be different for different meridians, which would be an obstacle to the adoption of the same calendar by all nations; indeed, when it is considered how pertinacious different nations are in reckoning geographical longitudes from their respective principal observatories, it cannot be supposed that they would all concur in making the commencement of the year to depend on the same meridian. We are therefore compelled to abandon the method pointed out by nature, and to recur to a mode of intercalating, which, though artificial, is regular and convenient. The simplest of all is that which Julius Cæsar introduced into the Roman calendar, and which consists in intercalating (p) one bissextile every four years. But if the short duration of life was sufficient to make the origin of the Egyptian years to deviate considerably from

the solstice or the equinox, it only required a small number of centuries to produce the same displacement in the origin of the Julian year. This renders a more complicated intercalation indispensable. In the eleventh century the Persians (*q*) adopted one remarkable for its accuracy. It consists in rendering the fourth year bissextile seven times successively, and to defer this change on the eighth time to the fifth year. This supposes that the tropical year is $365\frac{8}{33}$, which is greater than the year as determined by observations by 0,0001823. So that a great number of centuries is requisite to produce a sensible displacement in the origin of the civil year. The mode of intercalating adopted in the Gregorian calendar is less exact, but it furnishes greater facilities in reducing the years and centuries into days, which is one of the principal objects of the calendar. It consists in intercalating a bissextile every fourth year, the bissextile at the end of each century being suppressed, to reestablish it at the end of the fourth. The length of the year which this intercalation supposes is about $365, \frac{97}{400}$ days, or about 365,242500, which is greater than the true length by about $0,^{d}0002185$. But if, according to the analogy of this mode of intercalating, a bissextile is also suppressed every four thousand years, which would reduce the number of bissextiles in this interval to 969, the length of the year would be $365^{d}.\frac{969}{1000}$; or $365^{d},2422500$, which approaches so near to 365,2422419, which is the length as determined by observation, that the

difference may be rejected, particularly as there exists some slight uncertainty about the true length of the year, which besides is not rigorously constant.

The division of the year into twelve months is very ancient, and almost universal. Some nations have supposed that all the months are equal, and each to consist of thirty days, and they have completed the year by the addition of an adequate number of complementary days. Among other nations the entire year is comprized in the interval of twelve months, which are supposed to be unéqual. The system of months, each consisting of thirty days, leads naturally to their subdivision into three decads. This period enables us to find out with great facility how much of the month has lapsed, but at the end of the year the complementary days would derange the order of things appropriated to the different days of the decad, which must necessarily embarras the measures of governments. This inconvenience would be obviated by making use of a short period, equally independent of months and of years, such as the week, which from the most remote antiquity in which its origin is confounded, has uninterruptedly pervaded all nations, always constituting a part of the successive calenders of different people. It is very remarkable that it is identically the same over the entire earth, as well relatively to the denomination *(r)* of its days, which has been regulated by the most ancient system of astronomy, as also with respect to their correspondence to the

same physical instant. This is perhaps the most ancient and most incontrovertable monument of human attainments; it seems to indicate acommon origin from which they have been derived, but the astronomical system on which they were founded is a proof of their imperfection at this commencement.

An interval of one hundred years constitutes a century, which is the longest period ever employed in the measurement of time; for the interval which separates us from the most ancient known events does not require a longer period.

CHAP. IV.

Of the motions of the moon, its phases, and eclipses.

AFTER the sun, the moon, of all the heavenly bodies, is that which most interests us; its phases furnish a measure of time so remarkable, that it has been primitively made use of by all people. The moon, like the sun, has a proper motion from west to east; the duration of its sidereal revolution was $27^{d},321661423$, at the commencement of this century: it is not always the same, and the comparison of ancient with modern observations evinces incontrovertably an acceleration in the mean motion of the moon. This acceleration, though hardly sensible since the most ancient eclipse on record, will be developed in the progress of time. But will it go on always increasing, or will it cease to increase, and at length be changed into a retardation? This cannot be determined by observations, except after a very great number of ages. Fortunately, the discovery of its cause has anticipated them, and shewn us that it is periodical. At the commencement of this century, the mean angular distance of the moon from the vernal equinox, and reckoned from this equinox in the direction of the proper motion of this star, was 124,01321.

The moon moves in an elliptic orbit, of which the centre of the earth occupies one of the foci. Its radius vector traces about this point areas which are very nearly proportional to the times. The mean distance of this star from the earth being assumed equal to unity, the excentricity of its ellipse is 0,0548442, which gives the greatest equation of the centre equal (*s*) to 6,9854: it appears to be invariable. The lunar perigee has a direct motion, that is to say, in the direction of the proper motion of the sun, the duration of its sidereal revolution was, at the commencement of this century, $3232^{d},575343$, and its mean angular distance from the vernal equinox was $295°,68037$. Its motion is not uniform; it is retarded when that of the moon is accelerated.

The laws of the elliptic motion are very far from representing the observations of the moon; it is subject to a great number of inequalities, which have an evident connection with the position of the sun. We shall indicate the three principal.

The most considerable, and that which was first recognised is, what has been termed the *evection*. This inequality, which at its *maximum* amounts to $1°,4907$, is proportional to the sine of double the distance of the moon from the sun, minus the distance of the moon from its perigee. In the oppositions and conjunctions (*t*) of the moon with the sun, it is confounded with the equation of the centre, which it constantly diminishes. For this reason the ancient observers,

who only determined the elements of the lunar theory, in order to be able to predict the phenomena of the eclipses, found the equation of the centre of the moon less than the true equation, by the entire quantity of the evection.

Another great inequality is also observed in the lunar motions, which disappears in the oppositions and conjunctions of the moon, and also in those points where these two stars are distant from each other by a quarter of the circumference. It arrives at its *maximum*, which is 0°,6611, when their mutual distance is fifty degrees: hence it has been inferred that it is proportional to double of the angular distance of the moon from the sun. This inequality is termed (*t*) the *variation*: as it disappears in the eclipses, it could not have been recognized by the observation of these phenomena.

Finally, the motion of the moon is accelerated, when that of the sun is retarded, and conversely; hence arises an inequality which is denominated the *annual equation*, the law of which is precisely the same as that of the equation of the centre of the sun, only affected with a contrary sign. This inequality, which at its maximum (*u*) amounts to 0°,2074, is confounded with the equation of the centre of the sun in the eclipses. In the computation of the moment at which these phenomena occur, it is indifferent whether these two equations are considered separately, or whether the annual equation of the lunar theory is suppressed, in order to increase the equation of the centre of

the sun. This is the reason why the ancient astronomers assigned too great an excentricity to the orbit of the sun; while they assigned too small a one to the lunar orbit, in consequence of the *evection*.

This orbit is inclined to the ecliptic at an angle of 5°,7185: its points of intersection with the ecliptic, which are called the *nodes*, are not fixed in the heavens; they have a retrograde motion, or contrary to that of the sun; this motion is easily recognized by the succession of stars which the moon meets with when it traverses the ecliptic. The *ascending node* is that, in which the moon ascends above the ecliptic towards the north pole, and the *descending node* is that in which it descends below the ecliptic towards the south pole. The duration of a sidereal revolution of the nodes was at the commencement of this century 6793^d,39108, and the mean (*v*) distance of the ascending node from the vernal equinox, was 15°,46117, but the motion of the nodes is retarded from one century to another.

It is subject to several inequalities, of which the greatest is proportional to the sine of double the distance of the moon from the sun, and amounts at its *maximum* to 1°8102. The inclination of the orbit is likewise variable, its greatest inequality, which amounts to 0°,1627 at its *maximum*, is proportional to the cosine of the same angle on which the inequality of the motion of the nodes depends; however the mean inclination appears to be constant in different centuries, notwith-

standing the secular variations of the plane of the ecliptic.

The lunar orbit, and generally the orbits of the sun and of all the heavenly bodies, have no more a real existence than the parobolas described by projectiles at the surface of the earth. In order to represent the motion of a body in space, we conceive a line to pass through all the successive positions of its centre; this line is its orbit, of which the fixed or variable plane is that which passes through two consecutive positions of the body, and through the point about which it is supposed to move.

Instead of considering the motion of a body in this manner, we may in imagination project it on a fixed plane, and determine its curve of projection and height above this plane. This method, which is extremely simple, has been adopted by astronomers in the tables of the celestial motions.

The apparent diameter (*w*) of the moon changes in a manner analogous to the variations of the lunar motion; it is 5438″ at the greatest distance of the moon from the earth, and about 6207″ at the least distance. (*x*)

The same methods which were insufficient to determine the parallax of the sun, in consequence of its extreme smallness, have assigned 10661″ for the mean parallax of the moon; consequently at same distance at which the moon appears under an angle of 5823″, the earth would subtend an angle of 21332″; their diameters are therefore in the ratio of these numbers, or in the ratio of

three to eleven, very nearly; and the volume of the lunar globe is forty-nine times less than the volume of the earth.

The phases of the moon are one of the most striking phenomena of the heavens. When it extricates itself in the evening from the rays of the sun, it appears with a feeble crescent, which increases according as it elongates itself from the sun; and it becomes a perfect circle of light when it is in opposition with this star. When it afterwards approaches this star, its phases diminish in the same proportion as they had previously increased, until in the morning it is immersed in the sun's rays. The lunar crescent being always turned towards the sun, evidently indicates that it receives its light from that body, and the law of the variation of its phases, which increase nearly as the versed sine of the angular distance of the moon (y) from the sun, proves that it is spherical.

The recurrence of the phases depends on the excess of the motion of the moon above that of the sun, which excess (z) has been termed the *synodic* motion of the moon. The duration of the synodic revolution of this star, or the period of its mean conjunctions, is now about $29^{d},530588716$: it is to the tropical year very nearly in the ratio of 19 to 235, that is to say, nineteen solar years are equivalent to about two hundred and thirty-five lunar months.

The *syzygies* are the points of the orbit in which the moon is in opposition or conjunction

with the sun. In the first case the moon is said to be new, it is called full moon in the second. The *quadratures* are those points in which the moon is elongated from the sun one hundred or three hundred degrees, reckoning in the direction of its proper motion.

In those points, which are called the first and second quarters of the moon, we see the half of its illuminated hemisphere; strictly speaking, we see a little more; for when the exact half is presented to (*a*) us, the angular distance of the moon from the sun is a little less than one hundred degrees. At this instant the line which separates the illuminated from the darkened hemisphere, appears to be a right line, and the line drawn from the observer to the centre of the moon is perpendicular to the line which joins the centres of the sun and moon. Therefore in the triangle formed by the lines which join these centres and the eye of the observer, the angle at the moon is a right angle, and the angle at the observer is determined by observation, consequently the distance of the earth from the sun may be determined in parts of the distance of the earth from the moon. The difficulty of determining, with precision, the instant when the half of the disk of the moon is observed to be illuminated by the sun, renders this method not rigorously exact; we are indebted to it nevertheless, for the first just notions that have been formed (*b*) concerning the immense magnitude of the sun, and its great distance from the earth.

The explanation of the phases of the moon is connected with that of the eclipses, which, in times of ignorance, have been an object of terror to men, and of their curiosity in all ages. The moon can only be eclipsed by an opaque body, which deprives it of the light of the sun, and it is evident that this body is the earth, because an eclipse of the moon never happens except when it is in opposition, or when the earth is between this star and the sun. The terrestrial globe projects behind it a conical shadow, of which the axis is on the right line, which joins the centres of the sun and of the earth, and which terminates at the point where the apparent diameter of these two bodies, would be the same. Their diameters seen from the centre of the moon in opposition, and at its mean distance, are nearly 5920″ for the sun, and 21322′ for the earth: therefore the length of the cone of the earth's shadow is at least three times and a half greater than the distance of the moon from the earth, and its breadth at the points where it is traversed by the moon is about eight thirds of the diameter of the moon. The moon would be therefore eclipsed every time that it is in opposition to the sun, provided that the plane of its orbit coincided with the ecliptic; but in (c) consequence of the mutual inclination of these planes, the moon in its oppositions is frequently elevated above, or depressed below the cone of the earth's shadow, and it does not enter into it except when it is near to its nodes. If the entire disk of the moon is plunged in the

shadow of the earth, the eclipse of the moon is *total;* it is said to be *partial* if only a portion of this disk is obscured; and we may easily conceive that a greater or less proximity of the moon to its nodes, at the moment of opposition, may produce all the varieties *(d)* which are observed in these eclipses.

Each point of the surface of the moon, before it is eclipsed, loses successively the light of different parts of the sun's disk. Its brightness therefore diminishes gradually, and at the moment when it penetrates into the earth's shadow it is extinguished. The interval through which this diminution has place is termed the *penumbra*, the breadth of which is equal to the apparent diameter of the sun, as seen from the centre of the moon.

The mean duration of a revolution of the sun, with respect to the node of the moon's orbit, is about $346^d,619851$; it is to the duration of a synodic revolution of the moon, very nearly in the ratio of 223 to 19. Therefore, after a period of 223 lunar months, the sun and moon return to the same position relatively to the lunar orbit; the eclipses must consequently recur very nearly in the same order, this circumstance suggests a simple manner of predicting them, which was employed by the ancient astronomers. But the inequalities in the motions of the sun, and of the moon, ought to produce very sensible differences; besides the return of these two stars to the same position with respect to the node, in the interval of 223

months, is not rigorously exact; and the deviations which result, change at length the order of the eclipses which have been observed during one of these periods.

The circular form of the earth's shadow in the eclipses of the moon, indicated to the first astronomers that the figure of the earth was very nearly spherical; we shall see hereafter that the most exact method of determining the compression of the earth, is furnished by the great perfection to which the lunar theory has been brought.

It is solely in the conjunctions of the sun and of the moon, when this star, by being interposed between the sun and the earth, deprives us of the light of the sun, that the eclipses of the sun can be observed. Although the moon is incomparably smaller than the sun, yet on account of its proximity to the earth, its apparent diameter differs little from that of the sun; it even happens from the variations of these diameters, that they surpass each other alternately. Let us suppose the centres of the sun and moon to be on the same right line with the eye of the spectator, he will observe the sun to be eclipsed; and if the apparent diameter of the moon exceeds that of the sun, the eclipse will be total; but if this diameter be less, the observer will perceive a luminous ring, formed by that part of the sun which extends beyond the disk of the moon, and then the eclipse will be *annular*. If the centre of the moon does not exist in the right line drawn from the eye of the observer to the centre of the sun,

the moon can only eclipse a part of the sun's disk, and the eclipse will be partial. Thus the changes of distance of the sun and moon from the centre of the earth, combined with the greater or less proximity of the moon to its nodes, at the moment of its conjunctions, ought to produce very sensible changes in the eclipses of the sun. To these causes may be added the elevation of the moon above the horizon, which changes the magnitude of its apparent diameter, and which, by the effect of the lunar parallax, may so increase or diminish the apparent distance of the centres of these two stars, that of two observers at some distance from each other, the one may see an eclipse of the sun which will not be visible to the other. In this respect the eclipses of the sun differ from eclipses of the moon, which are the same to all places on the earth where the two stars are elevated above the horizon.

We often see the shadow of a cloud, borne along by the winds, to pass rapidly over the hills and planes, and to deprive the spectators in those places of the view of the sun, which is visible to those who are out of the reach of its influence: this is an exact representation of a total eclipse of the sun. We may perceive then about the disk of the moon a crown of pale light, which is probably the solar atmosphere; for its extent cannot correspond to that of the moon, because it has been ascertained, by eclipses of the fixed stars and of the sun, that this last atmosphere is almost insensible.

The atmosphere which may be supposed to surround the moon, inflects the rays of light towards the centre of this star; and if, as ought to be the case, the atmospherical strata are rarer in proportion as they are farther removed from the surface, these rays, according as they penetrate farther into it, ought to be more inflected, and should consequently describe a curve which is concave to its surface. Hence it appears that a spectator on the surface of the moon, would not cease to see the star till it was depressed below the horizon by an angle equal to the *horizontal refraction.* The rays which emanate from this star seen at the horizon, after having touched the surface of the moon, continue their route, describing a curve similar to that which they described in approaching the surface. Thus, a second spectator placed behind the moon, with respect to the star, would still continue to perceive it in consequence of the inflexion of its rays in the moon's atmosphere. The diameter of the moon is (*e*) not sensibly increased by the refraction of its atmosphere; therefore a star appears to be eclipsed later than if this atmosphere did not exist, and for the same reason it ceases to be eclipsed sooner, so that the effect of the atmosphere of the moon is principally apparent in the duration of the eclipses of the sun, and of the stars, by the moon. Precise and numerous observations have enabled us with difficulty to suspect its existence; and it has been ascertained that at the surface of the moon the horizontal refraction does not exceed four seconds.

This refraction at the surface of the earth is at least one thousand times greater; therefore the lunar atmosphere, if any such exists, is of an extreme rarity, greater even than that which can be produced on the surface of the earth by the best constructed air pumps. It may be inferred from this that no terrestrial animal could live or respire at the surface of the moon, and that if the moon be inhabited, it must be by animals of another species. There is ground for supposing that all is solid at its surface, for it appears in our powerful telescopes as an arid mass, on which some have thought they could perceive the effects, and even the explosions of volcanoes.

Bouguer has found by experiment that the light of the full moon (*f*) is three hundred thousand times more feeble than that of the sun; this is the reason why this light, collected in the focus of the most powerful mirrors, produces no sensible effect on the thermometer.

We may distinguish, especially near to the new moons, that part of the disk of the moon which is not illuminated by the sun. This feeble light, which has been termed the *lumiere cendree*, is supposed to be the effect of the light which the illuminated hemisphere of the earth reflects on the (*g*) moon; and that which confirms this supposition is the circumstance of this light being most sensible near to the new moon when the greatest part of this hemisphere is directed towards this star. In fact, it is evident that the earth exhibits to a spectator at the moon, phases

similar to those which the moon presents to us, but accompanied with a much stronger light, on account of the greater extent of the earth's surface.

The disk of the moon presents a great number of invariable spots, which have been carefully observed and described. They prove to us that this star always presents to us the same hemisphere; it revolves on its axis in a period equal to that of its revolution about the earth; for if a spectator be placed at the centre of the moon, supposed transparent, he will perceive the earth and his visual ray to revolve about him; and as this ray transverses always the same point of the moon's surface very nearly, it is evident that this point must revolve in the same time, and in the same direction as the earth about the spectator.

Nevertheless, a continued observation of the moon's disk indicates slight varieties in its appearances; spots are observed to approach and recede alternately from its borders; those which are very near to the borders, disappear and reappear successively, making periodical oscillations, which have been denominated *the libration of the moon.* In order to form a precise idea of the principal causes of this phenomenon, it should be considered that the disk of the moon, as seen from the centre of the earth, is terminated by the circumference of a circle of the lunar globe, which is perpendicular to its radius (*h*) vector, it is on the plane of this circle that the hemisphere of the moon, which is directed towards the earth, is pro-

jected, the appearances of which are connected with the motion of rotation of this star. If the moon had no motion of rotation, its radius vector would describe on its surface, in each lunar revolution, the circumference of a great circle, all the parts of which would be successively presented to us. But at the same time that the radius vector tends to describe this circumference, the lunar globe, by revolving, brings always very nearly the same point of its surface to this radius, and consequently the same hemisphere to the earth. The inequalities of the motion of the moon produce slight changes in its appearances; for as its motion of rotation does not participate in a sensible manner in these inequalities, it is variable relative to its radius vector, which thus meets its surface in different points; therefore the lunar globe makes, relatively to this radius, oscillations which correspond to the inequalities of its motion, and which alternately deprive us of and exhibit to us some parts of its surface.

Moreover, the lunar globe has another libration perpendicular to the preceding; in consequence of which the regions (*i*) which are situated near to the poles of rotation alternately disappear and reappear. In order to conceive this phenomenon, let the axis of rotation, be supposed perpendicular to the plane of the ecliptic. When the moon is in the ascending node, its two poles will be in the southern and northern extremities of the visible hemisphere. According as it ascends above the ecliptic, the north pole

and those parts which are contiguous to it, disappear, whilst more and more of those parts which border on the south pole are discovered, until the moon having attained its greatest northern latitude, recommences to descend towards the ecliptic. The preceding phenomena then takes place in a reverse order; and after that the moon, having arrived at the descending node, is depressed below the ecliptic, the north pole will present precisely the same phenomena as the south pole had previously exhibited.

The axis of rotation of the moon is not exactly perpendicular to the plane of the ecliptic, and its inclination produces appearances which may be conceived by supposing the moon to move on the plane itself of the ecliptic, in such a manner that its axis of rotation remains always parallel to itself. It is manifest that then each pole will be visible during one half of the revolution of the moon about the earth, and invisible during the other half, so that those parts which are contiguous to the poles will be alternately perceived and concealed.

Finally, the observer is not at the centre, but at the surface of the earth; it is the visible ray drawn from his eye to the centre of the moon, which determines the middle of the visible hemisphere; and on account of the lunar parallax, it is evident that this radius intersects the surface of the moon in points which depend on the height of the moon above the horizon.

All these causes produce only an apparent li-

bration in the lunar globe; they are purely optical, and do not affect the real motion of rotation. However, this motion may be subject to some small inequalities, though they are not sufficiently sensible to be discerned.

This is not the case with the variations of the plane of the lunar equator. From an attentive observation of the spots of the moon, DOMINICK CASINI inferred that the axis of this equator is not exactly perpendicular to the plane of the ecliptic, as had been supposed previous to his time, and also that its successive positions are not exactly parallel. This celebrated astronomer was led to the following remarkable result, one of his most splendid discoveries, and which contains the entire astronomical theory of the real libration of the moon. If through the centre of this star a plane be conceived to pass perpendicular to its axis of rotation, which plane coincides with that of its equator; if moreover we conceive a second plane to pass through the same centre parallel to that of the ecliptic, and a third plane, which is the plane of the lunar orbit, abstracting from the periodic inequalities of its inclination and of the nodes, these three planes have invariably a common intersection; the second situated between the two others, makes with the first an angle of about 1°,67, and with the third an angle of 5°,7155; consequently the intersections of the lunar equator with the ecliptic, or its nodes, coincide always with the mean nodes of the lunar orbit, and like them they have

a retrograde motion, of which the period is about $6793^d,391081$. In this interval, the two poles of the equator and of the lunar orbit describe small circles parallel to the ecliptic, its pole being comprised between them, so that these three poles exist always on the same great circle of the celestial sphere.

There are mountains on the surface of the moon, which rise to a considerable height; their shadows projected on the planes, form spots which vary with the position of the sun. At the edges of the illuminated part of the lunar disk, these mountains present the form of an indented border, which extends beyond the line of light by a quantity of which the measurement proves that their height is at least three thousand metres. From the direction of these shadows it has been inferred that the surface of the moon is intersected by deep cavities, similar to the basons of our seas. Finally, this surface seems to shew traces of volcanoes; and the formation of new spots, and the sparks which are observed in its obscure part appear to indicate (*k*) volcanoes in actual operation.

CHAP. V.

Of the Planets, and in particular of Mercury and of Venus.

In the midst of the infinite number of shining points which are spread over the celestial vault, and of which the relative position is very nearly constant, ten stars, always visible, except when they are immersed in the rays of the sun, move according to very complicated laws, the investigation of which constitutes one of the principal objects of astronomy. These stars, which have been denominated *planets*, are, Mercury, Venus, Mars, Jupiter, and Saturn, which have been known from the remotest antiquity, because they can be observed by the naked eye; and likewise Uranus, Ceres, Pallas, Juno, and Vesta, which have been recently discovered by means of the telescope. The two first planets never recede from the sun beyond certain limits; the others are elongated from it to all possible angular distances. The motions of all these bodies are comprehended in a zone of the celestial sphere, which is called the *zodiac*, the breadth of which is divided into two equal parts by the ecliptic.

The elongation of Mercury from the sun never exceeds thirty-two degrees: when it begins to ap-

pear in the evening, it is distinguished with difficulty in the rays of twilight; it extricates itself more and more on the succeeding days; and after it is elongated from the sun about twenty-five degrees, it returns towards him again. In this interval, the motion of Mercury, with respect to the stars, is direct; but when in approaching the sun, its distance from this star is about twenty degrees; it appears stationary, (*l*) and afterwards the motion becomes retrograde. Mercury still continues to approach the sun, and at length in the evening is again immersed in his rays. After continuing for some time invisible, it is again seen in the morning, emerging from the sun's rays, and receding from him. Its motion is still retrograde, as it was previous to the disappearance; but when the planet is twenty degrees elongated from the sun, it becomes again stationary, and afterwards resumes a direct motion; its elongation from the sun continues to increase till it becomes equal to twenty-five degrees, when the planet returns again, disappearing in the morning in the light of the dawn, and very soon after appearing in the evening, after which the same phenomena as before take place.

The extent of the greatest digressions of Mercury, or of his greatest deviations on each side of the sun, varies from eighteen to about thirty-two degrees. The duration of its total oscillations, (*m*) or of its return to the same position relatively to the sun, varies in like manner from one hundred and six, to one hundred and thirty days. The mean arc of retrogradation is about fifteen de-

grees, and its mean duration is twenty-three days; but these quantities differ considerably in different retrogradations. In general, the motion of Mercury is extremely complicated; it does not take place exactly in the plane of the ecliptic; some time this planet deviates five degrees from it.

A long series of observations was no doubt required to enable us to recognize the identity of the two stars, which were alternately observed in the morning, and in the evening, to recede from, and approach to the sun; but as the one was never seen until the other was invisible, it was at last concluded that it was the same planet which oscillated on each side of the sun.

The apparent diameter of Mercury is very variable, and its changes are evidently connected with its position relatively to the sun, and with the direction of its motion. It is a *minimum*, either when the planet in the morning is immersed in the sun's rays, or when in the evening it is disengaged from them. It is at its *maximum*, when in the evening it is immersed in these rays, or when it disengages itself from them in the morning. The mean apparent diameter is about 21″, 3.

Sometimes during the interval of its disappearing in the evening, and its re-appearing in the morning, the planet is seen projected in the form of a black spot on the disk of the sun, of which it describes a chord. It is recognized by its position, by its apparent diameter, and by its retrograde motion, being exactly those which it ought to have. These transits of Mercury are real annular eclipses

of the sun, which prove to us that this planet derives its light from the sun. When seen through a powerful telescope, it exhibits phases analagous to those of the moon, and, like to them, directed towards the sun, the variable extent of which, according to the position of the planet with respect to the sun, and according to the direction of its motion, throws great light on the nature of its orbit.

The planet Venus exhibits the same phenomena as Mercury, with this difference, that its phases are much more sensible, its oscillations more extensive, and their duration more considerable. The greatest digressions of Venus vary from about fifty to fifty-three degrees; and the mean duration of its oscillations, or of its return to the same position with respect to the sun, is about five hundred and eighty-four days. The retrogradation commences, or terminates, when the planet, approaching to the sun in the evening, or receding from him in the morning, is elongated from this star about thirty-two degrees. The arc of retrogradation is eighteen degrees very nearly, and its mean duration is forty-two days. Venus does not exactly move in the plane of the ecliptic, but sometimes deviates from it several degrees.

The durations of the passages of Venus over the disk, observed at places which are at considerable distances from each other on the surface of the earth, are very sensibly different, for the same cause which (*n*) makes the durations of the same eclipse of the sun different in different places. In

consequence of the parallax of this planet, different spectators refer it to different points of this disk, of which they observe it to describe chords of different lengths.

In the transit, which took place in 1769, the difference of its duration, as observed at Otaheite in the South Sea, and at Cajanibourgh in (*o*) Swedish Lapland, amounted to more than fifteen minutes. As these durations may be determined with very great exactness, their differences determine very accurately the parallax of Venus, and consequently its distance from the earth at the moment of its conjunction. A remarkable law, which we (*p*) shall explain at the end of our account of the discoveries which have made it known, connects this parallax with that of the sun and of all the planets; which circumstance renders these transits of peculiar importance in astronomy. After (*q*) succeeding each other in an interval of eight years, they do not recur again for more than a century, when they again succeed each other in the short interval of eight years, and so on continually. The two last transits happened on the fifth of June, 1761, and on the third of June 1769. Astronomers were sent to different places where the observations could be made under circumstances the most favourable for observing them, and from the result of their observations it has been concluded, that the parallax of the sun is about 26″,54 at its mean distance from the earth. The two next transits will take place on the eighth of December, 1874, and on the sixth of December, 1882.

The great variations of the apparent diameter of Venus, prove that its distance from the earth is very variable. This distance is least when it passes over the disk of the sun, and the apparent diameter is then about 189″: the mean magnitude of this diameter is, according to Arrago, about 52″,173.

From the motion of some spots observed on this planet, Dominique Cassini concluded that it revolves in an interval somewhat less than a day. Schroeter, by a continued observation of the variations of its horns, and by that of some luminous points near to the borders of those parts which are not illuminated, has confirmed this result, relative to which some doubts were entertained. He has determined the duration of its rotation to be 0^d,973; and he has found, with Dominique Cassini, that the equator of Venus makes a considerable angle with the ecliptic. (r) Finally, he has inferred from his observations that mountains of a considerable height exist on its surface; and from the law of the degradation of light in the passage from the obscure to the enlightened part, he inferred that the planet is surrounded by an extensive atmosphere, of which the refracting power does not differ much from that of the earth's atmosphere. The great difficulty of observing these phenomena even in the most powerful telescopes, makes it a matter of extreme delicacy to observe them in our climate: they demand every attention from those astronomers who, from their southern situation, enjoy a more favourable climate. But it is very

important, when the impressions are so feeble, to guard against the effects of imagination, which may considerably influence them; for then the interior images which it suggests, frequently modify and change those which the contemplation of objects produce.

Venus surpasses in brightness all the other stars and planets; it is sometimes so brilliant as to be seen in full daylight, and with the naked eye. This phenomenon, which depends on the return of the planet to the same position with respect to the sun, recurs in the interval of nineteen months very nearly, and its greatest brightness returns after an interval of eight years. Although it is of such frequent recurrence, it invariably excites surprise in the minds of the vulgar, who in their credulous ignorance, always suppose that it is connected with the most remarkable cotemporary events.

CHAP. VI.

Of Mars.

THE two planets which we have just considered, seem to accompany the sun, like satellites; and their mean motion about the earth is the same as that of this star. The other planets recede from the sun, to all possible angular distances, but their motions are so connected with that of the sun, that there can be no doubt of his influence on these motions.

Mars appears to us to move from west to east about the earth; the mean duration of his sidereal revolution is 687 days, very nearly; that of his synodic revolution, or of his return to the same position, relatively to the sun, is about 780 days. Its motion is very unequal; when it begins to be visible in the morning, the motion is direct and most rapid; it becomes gradually *(s)* slower, and vanishes when the angular distance of the planet from the sun is about 152°; afterwards the motion becomes retrograde, increasing in velocity till the moment of opposition of Mars with the sun. This velocity having then attained its *maximum*, diminishes, and again vanishes, when Mars in approaching to the sun, is distant from it by 152°. The motion after this becomes again

direct, having been retrograde for the space of seventy-three days, and in this interval the arc of retrogradation described by the planet is about eighteen degrees; continuing still to approach the sun, it finally is immersed in the evening in its rays. These remarkable phenomena are renewed at every opposition of Mars with the sun, but with a considerable difference as to the extent and duration of the retrogradations.

Mars does not move exactly in the plane of the ecliptic: it deviates sometimes several degrees from it. The variations of its apparent diameter are very great; it is about 19″,40 at the mean distance of the planet, and increases with the approach of the planet to opposition, where it amounts to 56″,43. At this time the parallax of Mars becomes sensible, and is nearly double of that of the sun. The same law which subsists between the parallaxes of the sun and of Venus, obtains also between the parallaxes of the sun and of Mars, and the observation of this last parallax had furnished a very near approximation of the solar parallax, before that it was determined with greater precision by the transits of Venus.

The disk of Mars is observed to change its form, and to become sensibly oval, according to its position relatively to the (*t*) sun. These phases shew that it receives its light from the sun. From the spots which are observed on its surface, it has been inferred that it revolves in a period of 1^{d},02733, on an axis inclined to the ecliptic in an angle of 66°,33. Its diameter in the (*u*) direction of the poles, is somewhat less

than the equatorial diameter. According to the measures of Arrago, these two diameters are in the ratio of 189 to 194, the preceding diameter being the mean between these two.

CHAP. VII.

Of Jupiter, and of his Satellites.

JUPITER moves from west to east in a period of $4332^{d},6$ very nearly, the duration of his synodic revolution is about 399^{d}. It is subject to inequalities similar to those of Mars. Previous to the opposition of this planet to the sun, and when its elongation from this star is almost one hundred and twenty-eight degrees, its motion becomes retrograde, its velocity increases till the moment of opposition, it then diminishes, vanishes, and finally resumes the direct state, when the distance of the planet as it approaches the sun, is only one hundred and twenty-eight degrees. The duration of this retrograde motion is one hundred and twenty-one days, and the arc of retrogradation is about eleven degrees; but there are very perceptible differences in the extent and in the durations of the different retrogradations of Jupiter. The motion of this planet does not exactly take place in the plane of the ecliptic; it sometimes deviates from it three or four degrees.

Several obscure belts have been observed on the surface of Jupiter; they are apparently parallel to each other, and to the ecliptic; other spots have also been observed, the motion of which has indi-

cated the rotation of this planet from west to east, on an axis very nearly perpendicular, to the plane of the ecliptic, and in a period (*v*) of $0^d,41377$. From the variations of some of these spots, and from the marked differences in the durations of the rotation, as inferred from their motions, it has been supposed that these spots do not adhere to the surface of Jupiter; they appear to be clouds which the winds transport with different velocities in a very agitated atmosphere.

Jupiter is, after Venus, the most brilliant of the planets, and even sometimes surpasses it in brightness. Its apparent diameter is the greatest possible in the oppositions, when it amounts to $141'',6$, its mean magnitude is $113'',4$, in the direction of the equator; but it is not the same in every direction. The planet is evidently compressed at the poles of rotation, and Arrago found, by very accurate measurement, that the polar is to the equatorial diameter, in the ratio of 167 to 177 very nearly.

Four small stars are observed to revolve about Jupiter, and to accompany this planet constantly. Their relative position changes every instant; they oscillate on each side of this planet, and it is by the extent of these oscillations, that their order is determined; we term the *first* satellite, that of which the oscillation is the least. They are sometimes observed to pass over the disk of Jupiter, and to project on it their shadow, which then describes a chord of the disk. It follows from this, that Jupiter and his satellites are opaque bodies,

illuminated by the sun ; and when they interpose between the sun and Jupiter, they produce real eclipses of the sun, precisely similar to those which the moon causes on the earth.

The shadow which Jupitar projects behind him, with respect to the sun, enables us to explain another phenomenon which the satillites present. They are observed frequently to disappear, although at a considerable distance from the disk of the planet: the third and fourth satillites reappear sometimes at the same side of this disk. These disappearances are altogether similar to the eclipses of the moon, and indeed all doubt on this head is removed by the concomitant circumstances. The satellites are always observed to disappear on the side of the disk of Jupiter which is opposite to the sun, and consequently on the same side with that to which the shadow of the cone is projected. The eclipse takes place nearest to the disk, when the planet is nearest to its opposition ; and finally, the duration of these eclipses corresponds exactly to the time which they should employ in traversing the cone of the shadow of Jupiter. Consequently these satellites move from west to east about this planet.

The observation of their eclipses furnish the most exact means of determining their motions. The durations of their periodical and synodical revolutions (*w*) about this planet are very precisely obtained, by comparing together eclipses which are separated from each other by considerable intervals, and which are observed near to

the opposition of this planet. By this means it has been ascertained that the motion of the satellites of Jupiter is almost circular and uniform, because this hypothesis satisfies very nearly those eclipses in which the planet is observed in the same position, with respect to the sun; therefore the position of these satellites, as seen from the centre of Jupiter, may be always determined.

Hence results a simple and tolerably exact method of comparing together the distances of Jupiter and the sun from the earth, a method which the antient astronomers did not possess; for the parallax of Jupiter, when nearest to us, is insensible even to the precision of modern observations; they had no data from which that distance could be judged of, except the duration of their revolutions, these planets being supposed to be most remote, the durations of whose revolutions were longest.

Let us suppose that the entire duration of an eclipse of the third satellite has been observed. At the middle of the eclipse, the satellite, as seen from the centre of Jupiter, is very nearly in opposition to the sun; therefore its sidereal position such as would be observed from this centre, and which it is easy to infer from the mean motion of Jupiter and of the satellite, is then the same as that of the centre of Jupiter seen from that of the sun. The position of the earth, as seen from the centre of the sun, may be had either from direct observation, or from the known motion of this star; therefore supposing a tri-

angle to be formed by lines joigning the centres of the earth, of the sun and of Jupiter, the angle at the sun will be obtained by what precedes; the angle at the earth will be given by direct observation; therefore at the middle of the eclipse the rectilinear distances of Jupiter from the earth and from the sun will be given in parts of the distance of the sun from the earth. It is found by this means, that when the apparent diameter of Jupiter is about 113″,4, he is at least five times more remote from us than the sun. The diameter of the earth would only appear under an angle of 10″,4, at the same distance; therefore the volume of Jupiter is at least one thousand times greater than that of the earth.

The apparent diameters of the satellites being insensible, their magnitudes cannot be measured exactly. An attempt has been made to estimate them, by the time which they take in penetrating into the shadow of the planet; but there is a great discordance in the observations which have been made to ascertain this circumstance, which arise from the different powers of the telescope, from the different degrees of perfection in the sight of the observers, from the state of the atmosphere, the heights of these satellites above the horizon, their apparent distance from Jupiter, and the change of the hemispheres which they present to us. The comparative brightness of the satellites is independent of the four first causes, which only produces a proportional change in their light; it may therefore furnish some infor-

mation concerning the return of the spots, which the rotation of these bodies ought to present successively to the earth, and consequently on the rotation itself. Herschell, who has been occupied with this delicate investigation, observed that they surpass each other successively in splendor, a circumstance that enables us to judge of the *maximum* and of the *minimum* of their light; and from a comparison of these maxima and minima, with the mutual positions of these stars, he has ascertained that they revolve on themselves, like the moon, in a period equal to the duration of their revolutions round Jupiter, a result which Maraldi had concluded to obtain in the case of the fourth satellite, from the returns of the same spot observed on his disk in its passages over the planet. The great distance of the heavenly bodies renders the phenomena which their surfaces present so extremely feeble, that they are reduced to slight variations of light, which cannot be perceived at the first view, and it is only after frequent experience in this kind of observation, that they become perceptible. But this means of supplying the imperfection of our organs, over which imagination has such control, ought to be employed with the greatest circumspection, to avoid being deceived respecting the existence of those varieties, and also lest we should be bewildered as to the causes on which they depend.

CHAP. VIII.

Of Saturn, of his Satellites, and of his ring.

SATURN revolves from west to east, in a period of 10759 days: the duration of his synodical revolution is 378 days. Its motion, which is performed very nearly in the plane of the ecliptic, is subject to inequalities similar to those of the motions of Mars and of Jupiter. Its retrograde motion commences and terminates when the distance of the planet from the sun before and after opposition is 121°: the duration of this retrogradation is about one hundred and thirty-nine days, and the arc of its retrogradation is about seven degrees. At the moment of opposition, the diameter of Saturn is at its *maximum*: its mean magnitude is about 50″.

Saturn presents a phenomenon which is *unique* in the system of the world. It is frequently observed in the middle of two small bodies which seem to adhere to it, the figure and magnitude of which are very variable; sometimes they are changed into a ring, which seems to surround the planet; at other times they disappear altogether, and Saturn then appears round like the other planets. By carefully following these remarkable appearances, and by combining them with the

positions of Saturn relatively to the sun and to the earth, Huygens ascertained that they are produced by a large and slender ring which surrounds the globe of Saturn, and is every where detached from it. This ring being inclined at an angle of 31°,85 to the plane of the ecliptic, always presents itself obliquely to the earth, in the form of an ellipse, of which the length when a maximum, is very nearly double the breadth. The ellipse becomes narrower in proportion as the visual ray drawn from Saturn to the earth, becomes less inclined to the plane of the ring, of which the more remote arc is at length concealed behind the planet, while the anterior arc is confounded with it; but its shadow, projected on the disk of Saturn, forms an obscure band, which being perceived in powerful telescopes, proves that Saturn and his ring are opaque bodies illuminated by the sun. We then only distinguish those parts of the rings which are extended on each side of Saturn; the breadth of these parts diminishes gradually, and they finally disappear, when the earth is in the plane of the ring, the thickness of which is imperceptible. The ring is likewise invisible when the sun being in its plane, only illuminates its thickness. It continues to be invisible as long as its plane is between the sun and earth, (*z*) and it reappears when the sun and earth are on the same side of this plane, in consequence of the respective motions of the sun and of Saturn.

As the plane of the ring meets the solar orbit at every semirevolution of Saturn; the phe-

nomena of the disappearance and reappearance recur very nearly after the interval fifteen years, but frequently under very different circumstances: two disappearances and two reappearances may occur in the same year, but never more.

During the disappearance of the ring, its thickness reflects to us the light of the sun, but in too small a quantity to be perceptible. However it may be conceived that by increasing the power of the telescope, it might be seen; and this is in fact what Herschell experienced during the last disappearance of the ring—which continued visible to him, when it had disappeared to other observers.

The inclination of the ring to the plane of the ecliptic is measured by the greatest opening which the ellipse presents to us: the position of its nodes with the plane of the ecliptic, is easily determined from the position of Saturn, when the appearance or disappearance of the ring, depends on the meeting of its plane with the earth. Therefore all the phenomena of this kind, which determine the same sidereal position of the nodes, take place when this plane meets the earth. When this plane passes through the sun, the position of its nodes determine that of Saturn, as seen from the centre of the sun, and then the rectilinear distance of Saturn from the earth, may be determined in the same manner as the distance of Jupiter is determined from the eclipses of his satellites. In the triangle formed by the three lines which join the centres of the sun, of Saturn, and of the

earth, the angles at the earth and sun are given, hence it is easy to conclude the distance of the sun from Saturn, in parts of the radius of the solar orbit. It is thus found that Saturn is about nine times and a half farther from us than the sun, when his apparent diameter is $50''$.

The apparent diameter of the ring, at its mean distance from the planet is, according to the accurate measures of Arrago, equal to $118'',58$; its apparent breadth is $17'',858$. Its surface is not continuous; a black band, which is concentrical with it, divides it into two parts, which appear to form two distinct rings, the breadth of the exterior being less than that of the interior. From several black bands which have been observed by some astronomers, it would appear, that there is a greater number of these rings. From the observation of some luminous spots of the ring, Herschell has ascertained that it revolves from west to east in a period of $0^{d},437$, about an axis which is perpendicular to its plane, and passing through the centre of Saturn.

Seven satellites have been observed to revolve round this planet from west to east, in orbits nearly circular. The six first move very nearly in the plane of the ring: the orbit of the seventh approaches more to the plane of the ecliptic. When this satellite is to the east of Saturn, its light becomes so feeble, that it is with very great difficulty perceived; this can only arise from the spots which cover the hemisphere which is presented to us. But in order that this phenomenon should

occur always in the same position, it is necessary that this satellite, (in this respect similar to the moon, and to the satellites of Jupiter,) should revolve on its own axis, in a period equal to that of its revolution about Saturn. Thus an equality between the periods of rotation and revolution appears to be a general law of the motion of the satellites.

The diameters of Saturn are not equal to each other. The diameter which is perpendicular to the plane of the ring, appears less by the eleventh part at least, than that which is situated in this plane. From a comparison of this compression with that of Jupiter, it may be inferred with great probability, that Saturn revolves rapidly about the least of his diameters, and that the ring revolves in the plane of his equator; this result has been confirmed by the direct observations of Herchell, which have indicated to him that the motion of this planet, like that of the other celestial bodies, is from west to east, and that its duration is 0,428, which differs very little from the duration of Jupiter's rotation. It is remarkable that this duration is very nearly the same, and less than half a day, for the two largest planets, while the planets which are less than them, revolve on their axes in the interval of a day very nearly.

Herchell has also observed on the surface of Saturn five belts, which are nearly parallel to his equator.

CHAP. IX.

Of Uranus and of his Satellites.

The planet Uranus escaped the observation of the ancient Astronomers on account of its minuteness. Flamstead at the end of the last century, Mayer and Le Monnier in the present, had already observed it as a small star. But it was not till 1781 that Herchell recognised its motion, and shortly after, by following this star carefully, he ascertained that it is an actual planet. Like to Mars, Jupiter and Saturn, Uranus moves from west to east about the earth. The duration of its sidereal revolution is about 30687 days; its motion, which takes place very nearly in the plane of the ecliptic, commences to be retrograde previous to its opposition, when the distance of the planet from the sun is 115°; its retrograde motion terminates, after opposition, when the elongation of the planet from the sun, as it approaches to this star, is 115°. The duration of its retrogradation is about 151 days, and the arc of retrogradation is four degrees.

If the distance of Uranus was to be estimated from the slowness of its motion, it should be on the confines of the planetary system. Its apparent diameter is very small, and hardly amounts

to twelve seconds. According to Herchel six satellites revolve about this planet in orbits almost circular, and very nearly perpendicular to the plan of the ecliptic. Telescopes of a very high magnifying power are required to enable us to perceive them ; two only, the second and fourth, have been recognized by other observers. The observations which Herchell has published relative to the four others, are not sufficiently numerous to enable us to determine the elements of their orbits, or even to be assured incontrovertably of their existence (*a*).

CHAP. X.

Of the Telescopic planets, Ceres, Pallas, Juno and Vesta.

THESE four planets are so minute, that they can be only perceived by means of very powerful telescopes. The first day of the present century is remarkable for the discovery which Piazzi made at Palermo of the planet Ceres. Pallas was recognized in 1802, by Olbers; Juno was discovered in 1803 by Harding; and lastly, Vesta was perceived in 1807 by Olbers. These stars, like the other planets, move from west to east; and like to them, they are alternately direct and retrograde. But in consequence of the short time which has elapsed since their discovery, we have not been able to determine with precision, the durations of their revolutions, and the laws of their motions. We only know that the durations of their sidereal revolutions differ little from each other; and that those of the three first are about four years and two thirds: the duration of the revolution of Vesta appears to be shorter by a year. Pallas deviates considerably more from the plane of the ecliptic than the other planets, so that in order to comprize its deviations, we should enlarge considerably the breath of the zodiac (*b*).

CHAP. XI.

Of the motion of the Planets about the sun.

HAD man restricted himself to a mere compilation of facts, the sciences would present nothing but a barren nomenclature, and a knowledge of the great laws of nature would never have been attained. It is from a comparison of facts with each other, by attentively considering their relations, and by this means reascending to phenomena, which are continually more and more extensive, that at length we have been enabled to discover these laws, which are continually impressed on the various effects which they produce. Then it is, that nature by revealing herself, shews how the infinite variety of phenomena which have been observed, may be traced up to a small number of causes, and thus enables us to determine antecedently those effects, which ought to be produced; and being assured that nothing will derange the connexion between causes and their effects, we can extend our thoughts forwards to the future, and the series of events which shall be developed in the course of time, will be presented to our view. It is solely in the theory of the system of the world, that the human mind has, by a long train of successful efforts, attained

to this eminence. The first hypothesis which was devised to explain the phenomena of the planetary motions, could only be an imperfect sketch of this theory, but by representing these phenomena in a very ingenious manner, it furnished the means of subjecting them to the calculus; and we shall now see, that by making this hypothesis to undergo the modifications which have been successively indicated by observation, it will be changed into the true system of the world.

The most remarkable of the planetary appearances is their change from a direct to a retrograde motion, a change which can only arise from two motions alternately conspiring together, and opposing their effects. The most natural hypothesis for explaining them, was that devised by the ancient philosophers, and which consisted in making the three superior planets to move in consequentia on epicycles, of which the centres describe circles in the same direction. It is manifest, that if the planet be supposed to exist in the lowest point of the epicycle, or that which is nearest to the earth, it has in this position a motion contrary to that of the epicycle, which is always moved parallel to itself; therefore if the first of these motions be supposed to predominate over the second, the apparent motion of the planet will be retrograde, and at its maximum; on the contrary, if the planet be situated at the most elevated point of its epicycle, the two motions conspire together, and the apparent motion is direct, and the greatest possible. In proceeding from the first to the se-

cond of these positions, the apparent motion of the planet continues to be retrograde; however, it constantly diminishes, till at length it vanishes, and then changes into a direct motion. It appears from observation, that the *maximum* of the retrograde motion obtains always at the moment of the opposition of the planet with the sun; it therefore follows that each epicycle is described in the time of a revolution of this star, and that the planet is at the lowest point, when it is in opposition to the sun. Hence we may see the reason why the apparent diameter of the planet is then at its *maximum*. With respect to the two inferior planets, which never deviate from the sun beyond certain limits, their alternate retrograde and direct motions may likewise be explained, on the hypothesis that they move in consequentia on epicyles, of which the centres describe, each year, circles about the earth in the same direction; and by supposing likewise that when the planet attains the lowest point of its epicycle, it is in conjunction with the sun. The preceding is the most ancient astronomical hypothesis, which being adopted and brought to perfection by Ptolemy, has been denominated from this astronomer.

The absolute magnitudes of the circles and of the epicycles are not indicated in this hypothesis: the phenomena only assign the relative magnitudes of the radii. In like manner Ptolemy did not attempt to investigate the respective distances of the planets from the earth; he only supposed

those superior planets to be farther from the earth, of which the times of revolution were the longest. He then placed the epicycle of Venus below the sun, and that of Mercury the lowest of all. In an hypothesis so indeterminate, it does not appear why the arcs of retrogradation of the superior planets are smaller, for those which are most remote; and why the moveable radii of the superior epicycles are parallel, to the radius vector of this star, and to the moveable radii of the inferior circles. This parallelism, which Kepler had already introduced into the hypothesis of Ptolemy, is clearly indicated by all observations of the motion of the planets, parallel and also in a direction perpendicular to the ecliptic. But if these epicycles and circles be supposed equal to the orbit of the sun, the cause of these phenomena become immediately apparent. It is easy to be satisfied that by such a modification of the preceding hypothesis, all the planets are made to revolve about the sun, which in his real or apparent motion about the earth carries along with it the centres of their orbits. A disposition of the planetary system so simple, leaves nothing undetermined, and clearly points out, the relations of the direct and retrograde motions of the planets, with the motion of the sun. It removes from the hypothesis of Ptolemy, the circles and epicycles which are described annually by these planets, and likewise those which he introduced in order to explain their motions perpendicular to the ecliptic. The relations which

this astronomer had determined to exist between the radii of the two inferior epicycles, and the radii of the circles described by their centres, express then the mean distances of the planets from the sun in parts of the mean distance of the sun from the earth; and the same relations being reversed for the superior planets, express their mean distances from the sun or from the earth. The simplicity of this hypothesis should of itself, induce us to admit it; but the observations which have been made by means of the telescope, remove all doubts on this subject.

It has been already observed, how the distance of Jupiter from the sun may be determined by the eclipses of the satellites of this planet, from which it appears that it describes about the sun, an orbit almost circular. We have also seen, that the appearances and disappearances of the ring of Saturn determine its distance from the earth to be about nine times and a half greater than the distance of the earth from the sun; and according to the determination of Ptolemy, this is very nearly the relation which obtains between the radius of the orbit of Saturn, and the radius of its epicycle; hence it follows that this epicycle is equal to the solar orbit, and that consequently Saturn describes very nearly a circle about the sun. From the phases which have been observed in the two inferior planets, it follows that they revolve about the sun. Let us for example follow the motion of Venus, and the variations of its apparent diameter and of its phases. When in

the morning it commences to extricate itself from the rays of the sun, it appears before the rising of this star, under the form of a crescent, and its apparent diameter is a *maximum;* it is then nearer to us than to the sun, and very nearly in conjunction with it. Its crescent increases, and its apparent diameter diminishes according as the planet elongates itself from the sun. When its angular distance from this star is about fifty degrees, it approaches towards it again, exhibiting to us more and more of its illuminated hemisphere: and the diminution of the apparent diameter continues to the'moment, that in the morning it is immersed in the sun's rays. At this intant, Venus appears to us full, and its apparent diameter is a *minimum ;* in this position it is farther from us than the sun. After continuing invisible for some time, this planet appears again in the evening, and reproduces in an inverted order, the phenomena which it exhibited previous to its disappearance. More and more of its illuminated hemisphere is averted from the earth: its phases diminish, and at the same time its apparent diameter increases with its increased elongation from the sun. When its angular distance from this star is about fifty degrees, it returns towards him: its phases continue to diminish, and its apparent diameter to increase, till it is again immersed in the rays of the sun. Sometimes in the interval between its disappearance in the evening, and its appearance in the morning, it is observed to move on the disk of the sun, in the form of a spot. It

is clear from these phenomena, that the sun is very nearly in the centre of the orbit of Venus, which it carries along with it, while it revolves about the earth. As Mercury exhibits phenomena which are similar to those of Venus, it follows that the sun is likewise in the centre of its orbit.

We are therefore conducted by the phenomena of the motions and of the phases of the planets, to this general result, namely, *that all these stars revolve about the sun, which in his real or apparent revolution about the earth, appears to carry with it the foci of their orbits.* It is remarkable that this result is derived from the hypothesis of Ptolemy, by supposing the solar orbit to be equal to the circles and epicycles which are described each year, in this hypothesis, which then ceases to be purely ideal, and only proper to represent to the imagination, the celestial motions. Instead of making the planets to revolve about imaginary centres, it places in the foci of their orbits, those great bodies which by their action can retain them in these orbits, and by this means it enables us to get a glimpse of the causes of the heavenly motions.

CHAP. XII.

Of the Comets.

STARS are frequently observed, which though at first scarcely perceptible, increase in magnitude and velocity, then diminish, and finally disappear. These stars, which are called *comets*, appear almost always accompanied with a nebulosity, which increasing, terminates sometimes in a tail of considerable length, and which must be extremely rare, as the stars are seen through its immense depth. The appearance of the comets followed by these long trains of light, had for a long time terrified nations, who are always affected with extraordinary events, of which they know not the causes. The light of science has dissipated these vain terrors which comets, eclipses, and many other phenomena excited in the ages of ignorance.

The comets participate, like the other stars, in the diurnal motion of the heavens; and this, combined with the smallness of their parallax, proves that they are not meteors generated in our atmosphere. Their proper motions are extremely complicated; they have place in every direction, and are not restricted, like the planets, to a motion from west to east, and in planes very little inclined to the ecliptic.

CHAP. XIII

Of the Stars, and of their motions.

The parallax of the stars is insensible; (*c*) their disks, viewed through the most powerful telescopes, are reduced to luminous points; in this respect, these stars differ from planets, of which the apparent magnitude (*d*) is increased by the magnifying power of the telescope. The smallness of the apparent diameter of the stars is particularly evinced by their rapid disappearance in their occultations by the moon, the time of which, not amounting to a second, indicates that this diameter is less than five seconds of a degree. The vivacity of the light of the most brilliant stars compared with the smallness of their apparent disk, induces us to think that they are much farther from us than these planets, and that they do not, like them, borrow their light from the sun, but are themselves luminous; and as the smallest stars are subject to the same motions as the most brilliant, and preserve the same position relatively to each other; it is extremely probable that the nature of all these stars is the same, and that they are so many luminous bodies of different magnitudes; and situated at greater or less distances from the limits of the solar system.

Periodical variations have been observed in the intensity of the light of several stars, which have been termed on that account *changeable.* Sometimes stars have been observed to appear suddenly, and then to vanish, after having shone with the most brilliant splendor. Such was the famous star observed in 1572 in the constellation of Cassiopeia. In a short time, it surpassed the most beautiful stars, and even Jupiter himself in brilliancy. Its light afterwards grew feeble, and in sixteen months after its discovery it disappeared, without having changed its place in the heavens. Its colour experienced considerable variations : it was first of a dazzling white, afterwards of a reddish yellow, and lastly, of a lead coloured white. What is the cause of these phenomena ? The extensive spots which the stars present to us periodically, in their revolution on their axes, in the same manner very nearly as the last satellite of Saturn, and perhaps the interposition of great opaque bodies which revolve about them, are sufficient to explain the periodical variations of the changeable stars. As to those stars which suddenly shine forth with a very vivid light, and then immediately disappear, it is extremely probable that great conflagrations, produced by extraordinary causes, take place on their surface ; and this conjecture is confirmed by their change of colour, which is analogous to that which is presented to us on the earth by those bodies, which are set on fire, and then gradually extinguished.

A white light of an irregular figure, (*d*) which

has been denominated the *milky way*, surrounds the heavens in the form of a zone. As a very great number of small stars has been discovered in it by means of the telescope, it is very probable that the milky way is nothing more than an assemblage of stars, which appear to us so near as to constitute an uninterrupted band of light. Small white spots, which are termed nebulæ, have also been observed in different parts of the heavens; several of which appear to be of the same nature as the milky way. When viewed through a telescope they likewise exhibit the union of a great mumber of stars; others only display a white and continuous light, perhaps on account of their great distance, which confounds the light of the stars which compose them. It is very probable that they are formed of a very rare nebulous matter, which is diffused in different masses in the heavenly regions, of which the successive condensation produces the nuclei, and all the varieties which they exhibit. The remarkable changes which have been observed in some of them, and particularly in the beautiful nebula of Orion, admit of a very easy explanation on this hypothesis, and render it extremely probable.

The immobility of the fixed stars with respect to each other, has determined astronomers to refer to them as to so many fixed points, the proper motions of the other heavenly bodies; but for this purpose it was necessary to classify them, in order that they might be recognized;

and it is with this view, that the heavens have been distributed into various groups of stars called constellations. It was likewise necessary to determine exactly the positions of the fixed stars on the celestial sphere, which has been accomplished in the following manner :

Let a great circle be conceived to pass through the two poles of the world, and through the centre of any star ; this circle, which is termed the circle of declination, is perpendicular to the equator. The arc of this circle, comprised between the equator and the centre of the star, measures its declination, which is *north* or *south*, according to the denomination of the pole, to which it is nearest.

As all the stars situated in the same parallel have the same declination, it was necessary to introduce a new element in order to determine their position. The arc of the equator, comprised between the circle of declination and the vernal equinox, has been selected for this purpose. This arc, reckoned from the equinox in the direction of the proper motion of the sun, *i. e.* from west to east, is termed the *right ascension*, consequently, the position of the stars is determined by their right ascension and declination.

The distance from the equator, or the right ascension, is determined by the meridian altitude of the star compared with the height of the pole. The determinations of its right ascenstion presented greatər difficulties to the antient astronomers, on account of the impossibility of compar-

ing directly the fixed stars with the sun. As the moon may be compared during the day with the sun, and during the night, with the fixed stars, they made use of it as an intermediate term, in order to measure the difference between the right ascension of the sun and of the fixed stars, having regard to the proper motions of the sun and moon, in the interval between the observations. The theory of the sun afterwards giving its right ascension, they inferred from it that of some of the principal stars, to which they compared the rest. It was by this means, that Hipparchus formed the first catalogue of fixed stars of which we have any knowledge. A considerable time after, this method was rendered much more precise, by employing, instead of the moon, the planet Venus, which is sometimes visible during the day, and of which during a short interval the motion is slower aud less unequal than the lunar motion. Now, that the important application of the pendulum to clocks, furnishes a very exact measure of time, we can determine directly, and with much greater precision than the ancient astronomers, the difference between the right ascenscion of the star and of the sun, by the interval of time which elapses between their transits over the meridian.

The position of the stars may be referred to the ecliptic in a similar manner, which is particularly useful in the theory of the moon and of the planets. A great circle is supposed to pass through the centre of the star, perpendicular to the plane of the ecliptic, which is called a circle of *latitude*. The arc of this circle comprised

between the ecliptic and the star, measures its latitude, which is north or south, according to the denomination of the pole situated at the same side of the ecliptic. The arc of the ecliptic comprised between the circle of latitude and the vernal equinox, reckoned from this equinox, in the direction of the sun's proper motion *i, e*, from west to east, is called the *longitude* of the star, the position of which is thus determined by its longitude and latitude. It may be easily conceived that the inclination of the ecliptic to the equator being known, the longitude and latitude of a star may be deduced from its observed right ascension and declination.

An interval of only a few years, was necessary to observe the variation of the fixed stars in right ascension and declination. It was very soon remarked that while they changed their position with respect to the equator, they preserved the same latitude, from which it may be inferred that the variations in right ascension and declination, arise solely from a motion common to these stars about the poles of the ecliptic. These variations might also be represented by supposing the stars immoveable, and by making the poles of the equator to move about those of the ecliptic. In this motion the inclination of the equator to the ecliptic remains constant, and its nodes or equinoxes regrade uniformly, at the rate of 154″,63 for each year. It has been already remarked that this retrogradation of the equinoxes, renders the tropical somewhat shorter than the sidereal

year. Thus the difference between the tropical and sidereal years, and the variations of the fixed stars in right ascension and declination, depend on this motion, by which the pole of the equator describes annually an arc of 154″,63 of a small circle of the celestial sphere parallel to the ecliptic. It is (*e*) in this, that the phenomenon known by the name of the precession of the equinoxes, consists.

The precision of modern astronomy, for which it is indebted to the application of telescopes, to astronomical instruments, and to that of the pendulum to clocks, has rendered perceptible, minute periodical variations in the inclination of the equator to the ecliptic, and in the precession of the equinoxes. Bradley, who discovered, and attentively followed them for several years, has observed their law, which may be geometrically represented in the following manner. Let the pole of the equator be supposed to move on the circumference of a small ellipse, a tangent to the celestial sphere, and of which the centre, which may be regarded as the mean pole of the equator, describes every year 154″,63 of the parallel to the ecliptic, on which it is situated. The greater axis of this ellipse, always in the plane of the circle of latitude, is equivalent to an arc of this great circle, equal to 59″,56; and the lesser axis is equivalent to an arc of this parallel, which is equal to 111″,30. The situation of the real pole of the equator on this ellipse is determined in the following manner: Suppose a small circle to be described in the plane of this ellipse, concentrical with it, and having its

diameter equal to the greater axis; conceive also a radius of this circle moved uniformly with a retrograde motion, so that this radius may coincide with that half of the greater axis which is nearest to the ecliptic, every time that the ascending node of the moon's orbit, coincides with the vernal equinox; and lastly, from the extremity of this moveable radius let fall a perpendicular on the greater axis of the ellipse, the point where this perpendicular intersects the circumference of the ellipse is the place of the true pole of the equator. This motion of the pole is termed *nutation.*

The fixed stars, in consequence of the motions which we have described, preserve an invariable position relatively to each other; but the illustrious observer to whom we are indebted for the discovery of the nutation, has discovered in all the stars a general periodical motion, which produces a slight change in their respective positions. In order to represent this motion, each star is supposed to describe annually a small circumference parallel to the ecliptic, of which the centre is the mean position of the star, and of which the diameter, as seen from the earth, subtends an angle of 125″, and that it moves on this circumference like the sun in his orbit, in such a manner however, that the sun is always more advanced than the star, by one hundred degrees; this circumference, projected on the surface of the heavens, appears under the form of an ellipse more or less flattened according to the height of the star above the ecliptic; the

lesser axis of the ellipse being to the greater axis, as the sine of this height is to the radius. Hence arise all the varieties of that periodical motion of the stars, which is called *abberration.*

Independently of those general motions, several stars have proper motions peculiar to themselves, very slow, but which the lapse of time has rendered sensible. They have been hitherto principally remarkable in Syrius and Arcturus, two of the most brilliant stars, but every thing induces us to think that in succeeding ages similar motions will be developed in the other stars.

CHAP. XIV.

Of the figure of the earth, of the variation of gravity at its surface, and of the decimal system of weights and measures.

LET us now descend from the heavens to the earth, and see what can be derived from observations relative to its dimensions and figure. It has been already observed that the earth is very nearly spherical: gravity being every where directed to the centre, retains bodies on its surface, although in places diametrically opposite, which are antipodes one to the other, they have directly contrary positions. The sky and the stars appear always above the earth; for elevation and depression are only relative terms with respect to the direction of gravity.

From the moment that man recognized the spherical form of the globe which he inhabits, he must have been anxious to measure its dimensions; it is therefore extremely probable that the first attempts to attain this object were made at a period long anterior to those of which history has preserved the record, and that they have been lost in the moral and physical changes which the earth has undergone. The relation which several measures of the most remote antiquity have to

each other, and to the terrestrial circumference, gives countenance to this conjecture, and seems to indicate not only that this length was very exactly known at a very ancient period, but that it has also served as the base of a complete system of measures, the vestiges of which have been found in Asia and in Egypt. Be this as it may, the first precise measure of the earth, of which we have any certain knowledge, is that which Picard executed in France towards the end of the seventeenth century, and which has been repeatedly verified. It is easy to conceive this operation. As we advance towards the north, the pole seems to be elevated more and more; the meridian heights of the stars situated towards the north increases, and that of the southern stars diminishes; some of them even become invisible. The notion of the curvature of the earth was no doubt suggested by observing these phenomena, which could not fail to attract the attention of men in the first age of society, when the return of the seasons was only distinguished by the rising and setting of the principal stars, compared with that of the sun. The elevation or the depression of the stars makes known the angles, which verticals raised at the extremities of the arc of the earth, which has been passed over, make at the point where they meet; for this angle is evidently equal to the difference of the meridian heights of the same star, *minus* the angle which the arc described would subtend at the centre of the star; and we are certain that this last angle is insensible. It

is then only necessary to measure this space. It would be a tedious and troublesome operation to apply our measures to so great an extent; it is much simpler to connect its extremities, by means of a series of triangles, with those of a base of twelve or fifteen thousand metres; and considering the precision with which the angles of these triangles may be determined, its length can be obtained very accurately. It is thus, that the arc of the terrestrial meridian which traverses France has been measured. The part of this arc of which the amplitude is the hundreth part of a right angle, and whose middle point corresponds to 50°, of altitude of the pole, is very nearly one hundred thousand metres.

Of all the re-entring figures, the spherical is the simplest, because it depends only on one element, namely, the magnitude of its radius. The natural inclination of the human mind to attribute that figure to objects, which it conceives with the greatest facility, disposed it to assign a spherical form to the earth. But the simplicity of nature should not be always regulated by that of our conceptions. Infinitely varied in her effects, nature is only simple in her causes, and her economy consists in producing a great number of phenomena, which are frequently very complicated, by means of a small number of general laws. The figure of the earth is the result of those laws, which modified by a thousand circumstances, might cause it to deviate sensibly from that of a sphere. Small variations, observed in the length

of the degrees in France, indicate these deviations; but the inevitable errors of observation left doubts on this interesting phenomenon; and the Academy of Sciences, in which this interesting question was anxiously discussed, judged with reason, that the difference of degrees, if it really existed, would be principally evinced in a comparison of the degrees at the equator and towards the poles. And academicians were sent even to the equator itself, where they found the degree of the meridian less than the degree of France. Other academicians travelled towards the north, where the degree was observed to be greater than the degree in France. Thus the increase of the degrees of the meridian, from the equator to the poles, was proved incontrovertably by these measures, from which it was concluded that the earth was not exactly spherical.

These celebrated voyages of the French Academecians having directed the attention of astronomers towards this object, new degrees of the meridian were measured in Italy, Germany, Africa, India and Pennsylvania. All these measures concur in indicating an increase in the degrees, from the equator to the poles.

The following table exhibits the values of the extreme degrees which have been measured, and also of the mean degree between the equator and the pole. The first was measured in Peru, by Bouguer and La Condamine. The length of the second has been inferred from the great operation which was recently executed, in order to deter-

mine the amplitude of the arc, which traverses France from Dunkirk to Perpignan, and which has been extended to the south, as far as Formentera. It was joined towards the north with the meridian of Greenwich, by connecting the sides of France with those of England, by means of a series of triangles. This immense arc, which comprises the seventh part of the distance of the pole from the equator, has been determined with the greatest precision. The astronomical and geodesical observations have been made with repeating circles. Two bases, each of which is more than twelve thousand metres, have been measured, the one near Melun, the other near to Perpignan, by a new process, which is free from all uncertainty; and what confirms the accuracy of these observations is, that the base of Pepignan concluded from that of Melun, by the chain of triangles which unites them, does not differ by a third of a metre from its actual measure, although the distance between those two places is upwards of nine hundred thousand metres.

In order to render this important observation as perfect as possible, the height of the pole, and the number of ocillations performed in a day by the same pendulum, have been observed on different points of this arc; from which the variations of the degrees and of gravity have been inferred. Thus this operation, the most accurate and extensive of the kind, which has been undertaken, will remain a monument of the state of arts and sciences in this enlightened age. Lastly,

the third degree was measured by M. Swanberg in Lapland.

Height of the pole.	*Length of the degree.*
0°,00	99523^{m},9.
50°,08	100004,3.
73°,71	100323,6.

The increase of the degrees of the meridian, according as the height of the pole increases, is even sensible in different parts of the great arc already mentioned. In fact let us compare its extreme points, and the Pantheon at Paris, which is one of the intermediate positions. It is found by means of observation,

	Height of the pole.	*Distances from Greenwich in the direction of the meridian.*
Greenwich	57,°19753	0^{m},0
Pantheon	54,°27431	292719,3
Formentera	42,°96178	1423636,1

The distance from Greenwich to the Pantheon, gives 100135^{m},2 for the length of the degree, of which the middle point corresponds to 55,°73592 of elevation of the pole; and from the distance of the Pantheon from Formentera, it is found that the length of a degree, the middle point of which corresponds to a latitude of 48,61804, is equal to 99970^{m},3, from which it follows that in the interval between these two points, the increment of a degree is 23^{m},167.

The ellipse being after the circle, the most simple of all the re-entring curves, the earth was considered as a solid of revolution formed by the

revolution of an ellipse about its lesser axis. Its compression in the direction of the poles, is a necessary consequence of the observed increase of the meridional degrees, from the equator to the poles. The radii of these degrees being in the direction of gravity, they are by the laws of the equilibrium of fluids, perpendicular to the surface of the seas with which the earth, is in a great measure covered. They do not terminate, as in a sphere, in the centre of the ellipsoid; they have neither the same direction, nor the same magnitude, as radii drawn from the centre to the surface, and which cut it obliquely every where except at the equator and at the poles. The point where two adjoining verticals situated in the same meridian meet, is the centre of a small terestrial arc comprized between them; if this arc was a right line, these verticals would be parallel, *i. e.* they would meet at an infinite distance; but in proportion as they are curved, they meet at a distance which is proportionally less as the curvature is greater; thus the extremity of the lesser axis being the point where the ellipse approaches most to a right line, the radius of a degree of the pole, and consequently the degree itself, is of its greatest length. It is the contrary at the extremity of the greater axis of the ellipse, *i. e.* at the equator, where the curvature being the greatest, the degree in the direction of the meridian is least of all. In proceeding from the second to the first of these extremes, the degrees continually increase; and if the compression of the ellipse is

inconsiderable, their increment is very nearly proportional to the square of the sine of the height of the pole above the horizon.

The excess of the equatorial axis, above that of the pole, assumed equal to unity, is termed the *compression* or *ellipticity* of the spheroid. The measure of two degrees in the direction of the meridian, is sufficient to determine it. A comparison of the arcs measured in France and Peru, which from their extent, their distance from each other, and from the accuracy and reputation of the observers, deserve the preference, makes the ellipticity of the terrestrial spheriod equal to $\frac{1}{311.7}$; the semiaxis major equal to 6376606^{m}, and the semiaxis minor is equal to 635625^{m}.

If the earth was elliptical, the same compression should be nearly obtained, from a comparison, two by two, of different measures of the terrestrial degrees; but their comparison gives, on this point, differences which it is difficult to ascribe solely to the errors of observations. It therefore appears that the earth differs sensibly from the ellipsoid. This difference is even indicated by the measures of different parts of the great arc of the meridian which traverses France; for it has been observed already, that the increment of its degrees is $23^{m},167$, which answers to an ellipticity of $\frac{1}{203}$, which more inconsiderable than the preceding ellipticity $\frac{1}{311.7}$; there is even reason to suppose that the two terrestrial hemispheres are not similar on each side of the

equator. The degree measured by La Caille at the Cape of Good Hope, where the height of the south pole is 37°,01, is (*f*) found to be equal to 100050^m,5; which is greater than that which was measured in Pennsylvania, where the height of the north pole is equal to 43°,56, the length of which was equal to 99789^m,1; it even exceeds the degree which was measured in France at an elevation of the pole equal to 50°, yet the degree at the Cape ought to be less than these degrees, if the earth was a regular solid of revolution formed of two similar hemispheres; every thing therefore leads us to think that this is not the case. But the considerable errors which new measures have frequently indicated in this kind of observation, ought to make us very cautious in the conclusions which we deduce from it, and to resolve to take all possible precautions to avoid for the future similar errors. Let us see then what is the nature of the terrestrial meridians, the earth being supposed to be any figure whatever.

The plane of the celestial meridian determined by astronomical observations, passes through the axis of the world and through the zenith of the observer; because this plane bisects the arcs of all lesser circles parallel to the equator, which are described by the stars above the horizon. All places of the earth, which have their zeniths in the circumference of this meridian, form the corresponding terrestrial meridian. Considering the immense distance of the stars, verticals elevated from each of these places may be supposed pa-

rallel to the plane of the celestial meridian; the terrestrial meridian may therefore *(g)* be defined to be that curve which is formed by the junction of the bases of all the verticals parallel to the plane of the celestial meridian. This curve lies altogether in this plane, when the earth is a solid of revolution; in every other case it deviates from it, and generally it is one of those lines which geometricians term *curves of double curvature.*

The terrestrial meridian is not exactly the line which determines trigonometrical measurements in the direction of the celestial meridian. The first side of the line which is measured, is a tangent to the surface of the earth, and parallel to the plane of the celestial meridian; if this side be extended till it meets a vertical indefinitely near to it, and if then this prolongation be bent to the base of vertical, the second side of the curve will be formed, and thus with all the others. The line thus traced is the shortest which can be drawn on the surface of the earth *(h)* between any two points assumed on this line; it does not lie in the plane of the celestial, and is not confounded with the terrestrial meridian, except in the case in which the earth is a solid of revolution; but the difference between the length of this line and that of the corresponding arc of the terrestrial meridian is so small that it may be neglected without any sensible error.

The figure of the earth being extremely complicated, it is important to multiply its measures

in every direction, and in as many places as possible. We may always at every point of its surface suppose an osculatory ellipse, which sensibly coincides with it for a small extent, about the point of osculation. Terrestrial arcs measured in the direction of the meridians, and of perpendiculars to the meridians, will make known the nature and position of this ellipsoid, which may not be a solid of revolution, and which varies sensibly at great distances.

Whatever be the nature of the terrestrial meridians, it is evident that as the degrees diminish from the poles to the equator, the earth is flattened in the direction (*i*) of the poles, *i. e.* that the axis of the earth is less than the diameter of the equator. In order to explain this, let us suppose that the earth is a solid of revolution; and let the radius of a degree at the north pole, and the series of those radii from the pole to the equator, which radii by hypothesis continually diminish, be supposed to be drawn, it is evident that these radii form by their consecutive intersections a curve, which at first touches the polar axis on the other side of the equator relatively to the north pole; it afterwards detaches itself from this axis, turning its convexity towards this axis, and continually raises itself towards the surface of the earth, until the radius of the meridional degree assumes a direction perpendicular to the primary direction; it is then in the plane of the equator. If the radius of the polar degree be supposed flexible, and that it involves successively the arcs of

the curve which have been just described, its extremity will describe the terrestrial meridian, and the part of it which is intercepted between the meridian and the curve will be the corresponding radius of the meridional degree. This curve is what Geometricians term the *evolute* of the meridian. Let the intersection of the diameter of the equator and of the polar axis be assumed for the present to be at the centre of the earth; the sum of the two tangents to the evolute of the meridian drawn from this centre, the first in the direction of the polar axis, and the second in the direction of the diameter of the equator, will be greater than the arc of the evolute comprised between them; but the radius drawn from the centre of the earth to the north pole is equal to the radius of the polar degree *minus* the first tangent; the semidiameter of the equator is equal to the radius of the meridional degree at the equator *plus* the second tangent; therefore the excess of the semidiameter of the equator above the terrestrial radius of the pole, is equal to the sum of those tangents, minus the excess of the radius of the polar degree above the radius of the meridional degree at the equator: this last excess is the arc itself of the evolute, which arc is less than the sum of the extreme tangents; consequently the excess of the semidiameter of the equator above the radius drawn from the centre of the earth to the north pole is positive. It can be proved in the same manner, that the excess of this same semidiameter of the equator above the radius drawn to the south pole is positive, therefore the

entire axis of the poles is less than the diameter of the equator, or what comes to the same thing, the earth is flattened in the direction of the poles.

Each part of the meridian being regarded as a very small arc of its osculatory circumference, it is easy to see that the radius drawn from the centre of the earth to the extremity of the arc, which is nearest to the pole, is less than the radius drawn from the same centre to the other extremity; hence it follows that the terrestrial radii continually increase from the poles to the equator, if, as all observations seem to indicate, the degrees of the meridian increase from the equator to the poles.

The difference of the radii of the meridional degrees at the poles and at the equator, is equal to the difference of the corresponding terrestrial radii *plus* the excess of (*k*) twice the evolute above the sum of the extreme tangents, which excess is evidently positive; thus, the degrees of the meridian increase from the equator to the poles in a greater ratio than that of the diminution of the terrestrial radii. It is evident that these demonstrations are equally applicable in the case in which the northern and southern hemispheres are not similar and equal, and it is easy to extend them to the case of the earth's not being a solid of revolution.

Curves have been constructed at the principal places in France, which lie on the meridian of the observatory, traced in the same manner as this

line, with this difference, that the first side, which is always a tangent to the surface of the earth, instead of being parallel to the plane of the celestial meridian of the observatory of Paris is perpendicular to it. It is by the length of these curves, and by the distances of the observatory from the points where they meet the meridian, that the positions of these places have been determined. This operation, the most useful which has been undertaken in geography, is a model which every enlightened nation should hasten to imitate, and which will very soon be extended to all Europe.

As the respective positions of places separated by vast seas cannot be fixed by geodesical observations, we must have recourse to celestial observations, in order to determine them. The knowledge of these positions is one of the greatest advantages which astronomy has procured. In order to arrive at it, the method which was made use of to form a catalogue of the fixed stars, was followed, by conceiving circles to be drawn on the surface of the earth corresponding to those which have been imagined on the celestial surface. Thus the axis of the celestial equator intersects the surface of the earth in two points diametrically opposite, which have respectively one of the poles of the world in their zenith, and which may be considered as the poles of the earth. The intersection of the plane of the celestial equator with this surface, is a circumference which may be regarded as the terrestrial equator; the intersections of all the planes of the celestial meridians with the same surface

are so many curved lines, which are reunited at the poles, and which are the corresponding terrestrial meridians, if the earth be a solid of revolution, which may be supposed in geography, without any sensible error. Finally, small circles traced on the earth parallel to the equator are terrestrial parallels; and that of any place whatever, corresponds to the celestial parallel which passes through its zenith.

The position of a place on the earth is determined by its distance from the equator, or by the arc of the terrestrial meridian comprised between its parallel and the equator, and by the angle which its meridian makes with the first meridian, of which the position is arbitrary, and to which all others are referred. Its distance from the equator depends on the angle comprized between its zenith and the celestial equator, and this angle is evidently equal to the height (l) of the pole above the horizon; this height is what in geography is termed *latitude*. The *longitude* is the angle which the meridian of a place makes with the first meridian; it is the arc of the equator contained between these two meridians. It is eastern or western, according as the place is to the east or west of the first meridian.

An observation of the height of the pole determines the latitude; the longitude is determined by means of a celestial phenomenon, which is observed simultaneously on the meridians of which the relative position is required. If the meridian of which the longitude is required is to the west

of that from which the longitude is reckoned, the sun will arrive sooner at the celestial meridian; if, for example, the angle formed by the terrestrial meridian be a fourth part of the circumference, the difference between the instants of noon, at those meridians, will be the fourth part of the day. Suppose, therefore, that a phenomenon is observed on each of them which occurs at the same physical instant for all places on the earth, such as the commencement or termination of an eclipse of the moon or of the satellites of Jupiter, the difference of the hours which the observers will reckon at the moment of the occurrence of the phenomenon, will be to an entire day as the angle formed by the inclination of the two meridians is to the circumference. Eclipses of the sun, and the occultations of the fixed stars by the moon, furnish the most exact means of obtaining the longitude, by the precision with which the commencement and termination of these phenomena may be observed; they do not in fact occur at the same physical instant at every place on the earth, but the elements of the lunar motions are sufficiently well known to enable us to make an exact allowance of this difference.

To determine the longitude of a place, it is not necessary that the celestial phenomenon should be observed at the same time on the first meridian. It is sufficient if it be observed under a meridian of which the position with respect to the first meridian is known. It is thus that by connecting meridians with each other, the respective

positions of the most distant points on the surface of the earth have been ascertained.

The longitudes and latitudes of a great number of places have been already determined by astronomical observations; considerable errors in the position and extent of countries a long time known, have been corrected: the position of those countries, which the interests of commerce, or the love of science have caused to be discovered, has been fixed; but though the voyages lately undertaken have added considerably to our geographical knowledge, much yet remains to be discovered. The interior of Africa, and that of New Holland, includes immense countries totally unknown: we have only uncertain, and frequently contradictory accounts concerning several others, of which geography hitherto abandoned to the hazard of conjecture, only waits for more accurate information from astronomy to fix and settle their position unalterably.

The longitude and latitude are not sufficient to determine the position of a place on the earth; besides these two horizontal coordinates, a vertical coordinate must be introduced, which expresses the elevation of the place above the level of the sea: this is the most useful application of the barometer; numerous and accurate observations with this instrument would throw the same light on the figure of the earth, (*m*) with respect to the comparative elevation of places, that has been already furnished by astronomy, on the other two dimensions.

It is principally to the navigator, when in the midst of the seas he has no other guide but the stars and his compass, that it is of consequence to know his position, that of the place for which he is bound, and of the shoals which he may meet in his passage. He may easily know his latitude by an observation of *(n)* the height of the stars: the fortunate inventions of the octant and of the repeating circle have rendered observations of this kind extremely accurate. But the celestial sphere, in consequence of its diurnal motion, presenting itself daily in very nearly the same manner to all the points of his parallel, it is difficult for the navigator to fix the *point* to which he corresponds. To supply the deficiency of celestial observations, he measures his velocity and the direction of his motion, thence he infers his progress in the direction of the parallels, and by a comparison of it with his observed latitude, he determines his longitude relatively to the place of his departure. The inaccuracy of this method subjects him to errors, which might become fatal when he abandons himself during the night to the winds near the shores and banks which, in his estimation, he believed himself at a considerable distance from. It is to secure him from these dangers that, as soon as the progress of arts and of astronomy led to the hope that methods might be devised to obtain the longitude at sea, commercial nations hastened to direct the views of scientific men and of artists to this important object, by powerful encouragements. Their expectations

have been realised by the invention of chronometers, and by the great accuracy with which the tables of the lunar motions have been constucted; two methods, good in themselves, and which are further improved by the mutual support which they confer on each other.

A chronometer, well regulated in a port, the situation of which is known, and which preserves the same rate when carried on board a vessel, would indicate, at every instant, the time which was reckoned in this port.

This hour being compared with that observed at sea, the relation of the difference of these hours to the entire day would be, as *(o)* has been already observed, that of the corresponding difference of longitude to the circumference. But it was difficult to obtain such watches; the irregular motion of the ship, the variations of temperature, and the inevitable friction which is extremely sensible in such delicate machines, were so many obstacles, all opposed to their accuracy. These have been fortunately surmounted; chronometers are now made which *(p)* for several months preserve a rate nearly uniform, and which thus furnish the simplest means of obtaining the longitude at sea; and as this method is always more exact as the time is shorter, during which these chronometers are employed, without verifying their rate, they are particularly useful in determining the position of places very near to each other. They have even, in this respect, some advantages over astronomical observations, the ac-

curacy of which is not increased by the proximity of the observers to each other.

The frequent recurrence of the eclipses of Jupiter's satellites would furnish an observer with an easy method of obtaining his longitude, if he could observe them at sea; but the endeavours which have been made to surmount the difficulties which the motion of the ship oppose to this kind of observations, have been hitherto fruitless; notwithstanding this, navigation and geography have derived considerable advantages from these eclipses, particularly from those of the first satellite, of which the commencement and termination can be accurately observed. The navigator employs them with success when he can land; indeed, it is necessary to know the hour at which the same eclipse which he observes would be seen upon a known meridian, since the difference of time, which is reckoned on these two meridians, gives the difference of longitudes; but from the great improvement which has been made in the tables of the first satellite in our time, the moment of the occurrence of these eclipses is given with a precision equal to that of observation itself.

The extreme difficulty of observing these eclipses at sea, has obliged us to have recourse to other celestial phenomena, among which the lunar motions are the only ones which can be made subservient to the determination of terrestrial longitudes. The position of the moon, such as it would be observed from the centre of the earth, may be easily inferred from the measure

of its angular distance from the sun and fixed stars : the tables of its motion then give the hour at the principal meridian when the same phenomenon is observed, and the navigator comparing the time which he reckons on board his ship at the moment of observation, determines his longitude by the difference of time.

To appreciate the accuracy of this method, it should be considered that from the errors of observation, the place, of the moon as determined by the observer, does not exactly correspond to the hour indicated by his chronometer ; and that in consequence of the errors of the tables this same place does not refer exactly to the corresponding hour which the sun indicates on the first meridian ; the difference of these hours would not therefore be such as would be furnished by an observation and tables rigorously correct. Suppose that the error produced by this difference is a minute. In this interval, forty minutes of the equator is passed over the meridian ; this is the corresponding error in the longitude of the vessel, and which is at the equator about forty thousand metres ; but it is less on the parallels, besides it may be diminished by multiplying observations of the lunar distances from the sun and stars, and repeating them during several days, in order that the errors of observation and of the tables may be mutually compensated and destroyed. It is obvious that the error in longitude corresponding to those of observation and of the tables are so much the less considerable, as the

motion of the celestial body is more rapid; thus observations made on the moon when in perigee, are in this respect, preferable to those made when the moon is in apogee. If the motion of the sun be employed, which is thirteen times slower than that of the moon, the errors in longitude will be about thirteen times as great; from hence it follows, that of all the celestial bodies the moon is the only one of which the motion is sufficiently rapid to be employed for the determination of the longitude at sea; we may consequently perceive of what great importance it is to render the tables as perfect as possible.

It is much to be desired that all the nations of Europe, instead of reckoning geographical longitudes from the meridian of their principal observatory, would concur in counting them from the same meridian, which being furnished by nature itself, might be easily found at all times. This agreement would introduce into their geography the same uniformity which their calendar and arithmetic present, a conformity which being extended to the various objects of their mutual relations, would constitute of these several nations but one immense family. Ptolemy caused his first meridian to pass through the Canaries, which were then the western limit of the known world. The reason of this selection no longer obtains, in consequence of the discovery of America. But one of these islands, presents one of the most remarkable points on the surface of the earth, in consequence of its great elevation and of its in-

sulation, namely, the summit of the peak of Teneriffe. We might with the Hollanders assume its meridian, from which to reckon terrestrial longitudes, by determining its position relatively to the principal observatories, by means of a great number of astronomical observations. But whether we agree or not as to a common meridian, it will be extremely useful for future ages to know accurately their position, with respect to some mountains which may be always recognized by their solidity and great elevation, such as Mount Blanc, which towers over the immense and imperishable woods of the Alpine regions.

A remarkable phenomenon, the knowledge of which we owe to astronomical voyages, is the variation of gravity at the surface of the earth. This singular power acts in the same place, on all bodies proportionally to their masses, and tends to impress on them equal velocities in equal times. It is impossible by means of a balance to ascertain these variations, because they equally affect the body weighed, and the weight to which it is compared; but they can be determined by a comparison of their weight with a constant force, such as the elasticity of the air at the same temperature. *(q)* Thus, by transporting to different places, a manometer filled with a column of air, which elevates by its tension a column of mercury in an interior tube, it is evident that an equilibrum must always subsist between the weight of this column and the elasticity of the air; its ele-

vation, when the temperature is given, will be reciprocally proportional to the force of gravity, the variations of which it consequently indicates. A very precise way of determining them is also furnished by observations of the pendulum; for it is obvious that its oscillations must be slower in those places where the gravity is less.

This instrument, the application of which to clocks is one of the principal causes of the progress of modern astronomy and geography, consists of a body suspended at the end of a thread or rod, moveable about a fixed point placed at the other extremity. The instrument is drawn a little from its vertical position, and being then remitted to the action of gravity, it makes small oscillations, which are very nearly of the same duration, notwithstanding the difference of the arcs described. This duration depends on the magnitude and figure of the suspended body, on the mass and length of the rod; but geometricians have found general rules to determine by observations of the compound pendulum, of any figure whatever, the length of a pendulum, the oscillations of which will be of a given duration, and in which the mass of the rod may be supposed nothing with respect to that of (r) the body, considered as an infinitely dense point. It is to this imaginary pendulum, termed the *simple pendulum*, that all the experiments of the pendulum made in different parts of the earth are referred.

Richer, sent in 1672 to Cayenne, by the Academy of Sciences, to make astronomical observa-

tions there, found that his clock regulated to mean time, at Paris, lost each day at Cayenne a perceptible quantity.

This interesting observation furnished the first direct proof of the diminution of gravity at the equator. It has been carefully repeated in a great number of places, taking into account the resistance of the air and the temperature. It follows from all the observed measures of a pendulum vibrating seconds, that it increases from the equator to the poles.

The length of the pendulum, which at the observatory of Paris makes one hundred thousand vibrations in a day, being assumed equal to unity, its length at the equator and at the level of the sea is equal to 0,99669, and in Lapland at an elevation of the pole equal to 74,22, it is observed to be 1,00137. Borda found by very exact and numerous experiments, that the length at the observatory at Paris which represented unity, was when reduced to a vacuum equal to 0,741887. From a repetition of these experiments by Biot and Mathieu, this length came out equal to 0,7419076, which differs very little from the preceding result. (*s*)

The increase in the length of the pendulum as we proceed from the equator to the poles, is even sensible on different points of the great arc of the meridian which traverses France, as will appear from an inspection of the following table, which gives the result of numerous accurate experiments made by Biot, Arrago and Mathieu.

Places.	*Height of the Pole.*	*Elevation above the sea.*	*Observed length of the pendulum which vibrates seconds.*
Fromentera	42°, 96	196^m	0^m,7412061
Bourdeaux	49, 82	0	0^m,7412615
Paris	54, 26	65	0, 7419076
Dunkirk	56, 67	0	0, 7420865

The observed lengths at Dunkirk and Bourdeaux give by the method of interpolations, 0,7416274 for the length of the pendulum which vibrates seconds on the coast of France, at the level of the sea, and at an elevation of the pole equal to fifty degrees. This length, and that of the meridional degree, the middle point of which corresponds to the same latitude, will enable us to recover our measures, if in the course of time they should be changed.

There is more regularity observed in the increase of the length of the pendulum, than in that of the meridional degrees: it deviates less from the ratio of the square of the sine of the pole's elevation; whether that its measurement being easier than that of degrees, it is less liable to error, or that the causes which disturb (*t*) the irregularity of the earth's form produce less effect on gravity. From comparison of all the observations which have been made on this subject, in different parts of the earth, it is found that if we assume for unity the length of the pendulum at the equator, its increase, as we proceed from the equator to the poles, is equal to the product of 0,005515 by the square of the sine of the latitude.

There has been likewise remarked by means of the pendulum, a small diminution of gravity on the summit of high mountains. Bouguer instituted a great number of experiments on this subject. At Peru he found that the force of gravity at the equator and at the level of the sea being expressed by unity, it is 0,999249 at Quito, which is elevated 2857^m above this level; and it is 998816 at Pinchincha, the elevation of which is 4744^m. This diminution of gravity (n) being sensible at elevations which are comparatively small with respect to the earth's radius, is a ground for supposing that it is considerable at great distances from the centre of the earth.

The observations of the pendulum furnishing a length which is invariable, and easy to be recovered at all times, has suggested the idea of employing it as an universal measure. The prodigious number of measures in use, not only among different people, but in the same nation; their whimsical divisions, inconvenient for calculation, and the difficulty of knowing and comparing them; finally, the embarrassments and frauds which they produce in commerce, cannot be observed without acknowledging that the adoption of a system of measures, of which the uniform divisions are easily subjected to calculation, and which are derived in a manner the least arbitrary, from a fundamental measure, indicated by nature itself, would be one of the most important services which any government could confer on society. A nation which would originate such a system of measures, would combine the advan-

tage of gathering the first fruits of it with that of seeing its example followed by other nations, of which it would thus become the benefactor; for the slow but irresistible empire of reason predominates at length over all national jealousies, and surmounts all the obstacles which oppose themselves to an advantage, which would be universally felt. Such were the reasons that determined the Constituent Assembly, to charge the Academy of Sciences with this important object. The new system of weights and measures is the result of the labours of a committee appointed by them, seconded by the zeal and abilities of several members of the national representation.

The identity of the decimal calculus with that of integral numbers, leaves no doubt as to the advantages of dividing every kind of measure into decimal parts. To be convinced of this, it is only necessary to compare the difficulties of complicated divisions and multiplications, with the facility by which the same operations are performed on integral numbers, which facility may be increased by logarithms, the use of which might be rendered very popular by simple and cheap instruments. Indeed our Arithmetical scale is not divisible by three and four, two divisors which, from their great simplicity, are (*v*) of very frequent occurrence. This advantage would be secured by the addition of two new characters. But such a marked alteration would be inevitably rejected, together with the system of measures which would have been conformed to it. The duodecimal scale would be

also subject to the additional inconvenience of requiring us to remember the binary products of the eleven first numbers, which surpasses the ordinary compass of the memory, to which the decimal scale is well adapted; lastly we could not retain the advantage which probably gave rise to our arithmetic, namely, that of making use of our fingers in reckoning. The academy therefore, did not hesitate in adopting the decimal division; and to render the entire system of measures uniform, it was resolved that they should all be derived from the same lineal measure, and from its decimal divisions. The question was thus reduced to the choice of this universal measure, which was denominated the *metre*.

The length of the pendulum, and that of the meridian, are the two principal means furnished by nature itself to fix the unity of linear measures. Both being independent of moral revolutions, they cannot experience a sensible alteration except by very great changes in the physical constitution of the earth. The first means, though easily applied, is notwithstanding subject to the inconvenience of making the measure of distance to depend on two elements which are heterogeneous to it, namely, gravity and time, the measure of which last is arbitrary; and as it is divided sexagesimally, it cannot be admitted as the foundation of a system of decimal measures. The second means was therefore selected, which appears to have been employed in the remotest antiquity; so natural is it for man to compare itinerary measures with the

dimensions of the globe itself which he inhabits; so that in travelling he may know by the mere denomination of the space he has passed over, the relation of that space to the entire circuit of the earth. There is also the additional advantage of making nautical and celestial measures to correspond. The navigator has frequent occasion to determine the one by the other, the distance he has traversed, and the celestial arc included between the zeniths of the places of his departure and arrival; it is therefore of consequence that one of these measures should be the expression of the other, by nearly the difference of their unities. But for this purpose, the fundamental unity of linear measures should be an aliquot part of the terrestrial meridian, which corresponds to one of the divisions of the circumference. Thus the choice of the metre was reduced to that of the unity of angles.

The right angle is the limit of the inclination of a line to a plane, and of the elevation of objects above the horizon; besides it is in the first quadrant of the circumference that the sines are formed, and generally all the lines which are employed in trigonometry, of which the proportions to the radius have been reduced into tables; it was therefore natural to assume the right angle as the unity of angles, and the quarter of the circumference for the unity of their measures. It is divided into decimal parts, and in order to obtain corresponding measures on the earth, the quarter of the terrestrial meridian has been divided into the same

parts, which had been done at a very ancient period; for the measure of the earth mentioned by Aristotle, the origin of which is unknown, assigns a hundred thousand stadia to the quarter of the meridian. It was then only necessary to obtain its exact length. Here two questions present themselves to be resolved. What is the proportion of an arc of the meridian measured at a given latitude, to the entire circumference? Are all the meridians similar? In the most natural hypotheses on the constitution of the terrestrial spheroid, the difference of the meridians is insensible, and the decimal degree of the middle point answering to the fiftieth degree of latitude, is the hundreth part of the quarter of the meridian. The error of these hypotheses can only influence geographical distances, where it is of no consequence. The length of the quarter of the meridian may therefore be conclnded from that of the arc which traverses France from Dunkirk to the Pyrenees, and which was measured in 1740, by the French Academicians. But as a (x) new measure of a greater arc, in which more accurate methods were employed, would excite an interest in favour of the new system of measures calculated to extend its utility, it was resolved to measure the arc of the terrestrial meridian contained between Dunkirk and Barcelona. This great arc extended as far south as Formentera, and to the north as far as the parallel of Greenwich, and of which its point of bisection, corresponds very nearly to the mean parallel between the Pole and the Equator,

has given for the length of the quarter of the meridian 5130740 toises.

The ten millioneth part of this length was taken for the metre or the unity of linear measures. The decimal above this was too great, and the decimal below it was too small, and the metre, the length of which is 0,513074 toises, supplies advantageously the place of the toise and ell, which were two of our measures in most common use.

All the measures are derived from the metre, in the simplest possible manner; the linear measures are decimal multiplies and sub-multiplies of it.

The unity of the measure of capacity is the cube of the tenth of a metre; it is called *litre.* The unity of the superficial measure of land is a square, the side of which is ten metres; it is called *are.*

A *stere* is a volume of fire-wood, equal to a cubic metre.

The unity of weight, which is termed *gramme,* is the absolute weight of the cube of a millioneth part of a metre of distilled water, when at its *maximum* of density. By a remarkable peculiarity, this *maximum* does not correspond to the freezing point, but is above it by about four degrees of the thermometer. Water, as it falls below this temperature, again dilates, and thus prepares itself for that increase of volume, which it undergoes in its passage from the fluid to the solid state. Water has been selected as being one of the most homogeneous substances, and which may be easily

reduced to a state of purity. Le Ferre Gineau has determined the *gramme* by a long series of delicate experiments on a hollow cylinder of brass, the volume of which he measured with extreme care; the result of these experiments is, that the livre being supposed equal to the twenty-fifth part of the pile of fifty marcs, which is preserved at the mint of Paris, is to the gramme in the ratio of 489,5058 to unity. The weight of a thousand grammes, which is denominated the *kilogramme or decimal livre*, is consequently equal to the livre, the weight of the marc multiplied by 2,04288.

In order to preserve the measures of length, and the unity of weights, standards of the metre and of the kilogramme, executed under the immediate superintendence of the committee to whom the determination of these measures was intrusted, and verified by them, were deposited in the national archives, and at the observatory of Paris. The standards of the metre do not represent it, except at a definite temperature. The temperature of melting ice was selected as being the most invariable, and independent of the modifications of the atmosphere. The standards of the kilogramme do not represent its weight, except in a vacuum, in which case the pressure of the atmosphere is insensible. In order to be able to recover the metre at all times, without having recourse to the measure of the great arc which furnished it, it was necessary to determine its relation to the length of the pendulum which

vibrates seconds ; this has been effected by Borda in the most accurate manner.

As there was necessarily a constant comparison of all these measures with the livre in money, it was particularly important to divide it into decimal parts. Its unity has been denominated the silver franc, its tenth part, *decime*, its hundreth *centieme*. The values of golden pieces of money, of gold and brass, have been referred to the franc.

In order to facilitate the calculation of the fine gold and silver contained in pieces of money, the alloy was fixed at the tenth part of their weight, and that of the franc has been made equal to five grammes. Thus the franc being an exact multiple of the unity of weights, it can be made use of in weighing bodies, which is extremely useful in commerce.

Finally, the uniformity of the whole system of weights and measures required that the day should be divided into ten hours, the hour into one hundred minutes, and the minute into one hundred seconds. This division of the day, which will be indispensable to astronomers, is of less consequence in civil life, where there is little occasion to imploy time as a multiplier and divisor. The difficulty of adapting it to watches and clocks, and our commercial relations with foreigners in the sale of watches, will suspend its application indefinitely. We may however be assured, that at length the decimal division of the day will supersede its present division, which differs too much

from the division of the other measures not to be abandoned.

Such is the new system of weights and measures, presented by the Academy to the National Convention, which immediately adopted it. This system, founded on the measure of the terrestrial meridians, corresponds equally to all nations. It has no other relation with France than what is furnished by the arc of the meridian which traverses it. But the position of this arc is so advantageous, that if the learned of all nations had combined to fix an universal measure, they would have selected it. To multiply the advantages of this system, and to render it useful to the entire world, the French government invited foreign powers to participate in an object of such general interest: many have sent eminent men of science to Paris, who, in conjunction with the committee of the National Institute, have determined by a discussion of observations and experiments, the fundamental unites of weights and lengths; so that the determination of these unites may be considered as a work common to the learned who have assembled there, and to the people of whom they are the representatives. It is therefore permitted to hope, that one day this system, which reduces all measures and their computations to the scale, and to the simplest operations of the decimal arithmetic, will be as universally adopted as the system of numeration of which it is the completion, and which, without doubt, had to surmount the same obstacles which prejudices

and long established habit oppose to the introduction of the new measures; but when once introduced, these measures will be maintained by this same power which, combined with that of reason, secures to human institutions an eternal duration.

CHAP. XV.

Of the flux and reflux of the sea, and of the daily variation of its figure.

ALTHOUGH the earth, and the fluids which are diffused over it, must long since have assumed the state which corresponds to the equilibrium of the forces which actuate them, nevertheless, the figure of the sea changes every instant of the day, by regular and periodical oscillations, which are denominated, *the ebbing and flowing of the sea.* It is a circumstance truly astonishing to behold, in calm serene weather, the intense agitation of this great fluid mass, of which the waves break with violence against the shores. This phenomenon gives rise to reflexions, and excites a strong desire to penetrate the cause. But in order that we may not be mislead by vague hypotheses, it is necessary previously to know the laws of this phenomenon, and to follow it in all its details. As a thousand accidental causes may alter the regularity of these phenomena, it is necessary to consider at once a great number of observations, in order that the effects of transient causes, mutually compensating each other, the mean results may only indicate the regular and constant effects. It is likewise necessary, by a judicious combination of observations,

to make each of these effects which we wish to determine, as conspicuous as possible. But this is not sufficient. The results of observations being always liable to error. it is necessary to know the probability that these errors are confined within given limits. Indeed it is evident, that for the same probability, these limits are more restricted, as the observations are more numerous; and this is the cause why observers have been at all times anxious to multiply the number of experiments and observations. But the degree of accuracy of the results is not indicated by this general impression; it does not make known the number of observations necessary to obtain a determinate probability. Sometimes even, it has induced us to investigate the cause of phenomena which arose from mere chance. It is by means of the calculus of probabilities alone that we are enabled to appreciate these objects, which renders its application in physical and moral sciences of the greatest importance.

At the request of the Academy of Science, a great number of observations were made in the beginning of the last century, in our harbours: they were continued every day at Brest during six successive years. The situation of this port is peculiarly favourable to this kind of observations. It communicates with the sea by a vast and long canal, at the extremity of which this port has been constructed. The irregularities in the motion of the sea, are consequently much diminished when they arrive at this port; just as the oscillations

which the motion of a vessel impresses on a column of mercury in the barometer, are considerably lessened by the contraction of the tube of this instrument. Moreover, the tides being very sensible at Brest, the accidental variations constitute but a very inconsiderable part of them; and if we particularly consider, as I have done, the excess of the high water over the preceding and subsequent low water, it will appear that the winds, which are the principal cause of the irregularities in the motion of the sea, have very little influence on the results; because if they raise the high water, they elevate very nearly as much the preceding and subsequent low water; so that a very great regularity has been observed in these results, considering the fewness of the observations which have been made. Struck by this regularity, I requested the government to order a new series of observations to be made in the harbour of Brest, during the entire period of the motion of the nodes of the lunar orbit. This has been accordingly done; they commenced in the year 1806, and have been uninterruptedly continued each successive day. All these observations being discussed, in the manner I previously made mention of, the following results have been obtained respecting which there cannot remain any doubt.

The sea rises and falls twice in the interval of time comprehended between two consecutive returns of the moon to the meridian, above the horizon. The mean interval of these returns is $1^d,035050$,; thus, the interval between two con-

secutive high tides is $0^{d},517525$, so that there are some solar days in which only one high tide can be observed. The moment of low water very nearly divides the extremities of this interval equally at Brest, the sea is longer rising than falling by above nine or ten minutes. Similar to (*a*) all magnitudes, which are susceptible of a *maximum* or a *minimum*, the increase or diminution of the tide near to these limits is proportional to the square of the time elapsed, since the moments of high or low water.

The elevation of the sea at high tide is not always the same; it varies every day, and its variations are evidently connected with the phases of the moon. It is greatest about the time of full or of new moon; it then diminishes and becomes least near to the time of quadrature. The highest tide at Brest does not take place exactly the day of the syzygy, but a day and a half later, so that if the syzygy happens at the moment of high tide, the greatest tide is the third that follows. In like manner, if the quadrature happens at the moment of high water, the third tide which follows will be the least. This phenomenon is observed to be very nearly the same in all the ports of France, although the hours of high and low water are very different.

The greater the elevation of the sea at high water, the more will it fall at the low water which succeeds it. A *total tide* is termed half the sum of the heights of two consecutive high waters, above the level of the intermediate low

water. The mean value of this total high water at Brest, at its maximum near to the syzygies, and when the sun and moon are in the equator, and at their mean distances from the earth, is about five metres and a half. In the same circumstances it is less by one half in the quadratures.

From an attentive consideration of these results, it appears that the number of high waters being equal to the number of passages of the moon over the upper or inferior meridian, this star has the principal influence on the tides; but from the circumstance of the tides in the quadratures, being fuller than those in syzygies, it follows that the sun also influences this phenomenon, and in some measures modifies the effect of the moon's influence. It is natural to think that each of these influences, if they existed separately, would produce a system of tides, of which the period would be the same as that of the respective stars over the meridian, and that from the combination of the two systems, there should arise a compound tide, in which the lunar high water would correspond to the solar high water near to the syzygies, and to the solar low water near to the quadratures. The declinations of the sun and of the moon have a remarkable influence on the tides; they diminish the total high waters of the syzygies and of the quadratures; they increase by the same quantity the total high waters of the solstices. Thus the received opinion that the tides are greatest in the equinoctial syzygies, is confirmed by an exact discussion of a

great number of observations. However, several philosophers, and especially Lalande, have questioned the truth of this observation, because that near to some solstices the sea rises to a considerable height. It is here that the calculus of probabilities is of such importance in enabling us to decide this important question in the theory of the tides. It has been found by applying this calculus to the observations, that the superiority of the syzygial equinoctial tides and of the solstitial sides in quadratures is indicated, with a probability much greater than that on which most of the facts respecting which there exists no doubt, rest.

The distance of the moon from the earth influences, in a very perceptible manner, the magnitude of the high water. All other circumstances being the same, they increase and diminish with the diameter and lunar parallax, but in a greater ratio. The variations of the distance of the sun from the earth, influences the tides in a similar manner, but in a much less degree.

It is principally near the *maxima* and *minima* of the total tides, that it is interesting to know the law of their variation. We have seen that the moment of their maximum at Brest follows the time of the occurrence of the syzygy by a day and a half. The diminution of the total tides which are near to it, is proportional to the square of the time which has elapsed from that instant, to that of the intermediate low water, to which the total tide is referred. Near the instant of the *minimum*, which follows the quadrature by a day and a half,

the increment of the total tide is proportional to the square of the time which has elapsed since this instant; it is very nearly double of the diminution of the total tides near to their *maximum*.

The declinations of the sun and of the moon sensibly influence these variations; the diminution of the tides near the syzygies of the solstices is only about three fifths of the corresponding diminution near the syzygies of the equinoxes; the increment of the tides near to the quadratures is twice greater in the equinoxes than in the solstices. But the effect of the different distances of the moon from the earth is still more considerable, than that of the declinations. The diminution of the syzygial high waters is nearly three times greater near to the lunar perigee, than it is near to its apogee.

A small difference has been observed between the morning and evening tides, which must depend on the declinations of the sun and of the moon, as the differences disappear when these stars are in the equator. In order to recognize them, we should compare the tides of the first and of the second day after the syzygy or the quadrature; the tides being then very near to the maximum or the minimum, vary very little from one day to another, which enables us easily to observe the difference between two tides of the same day. It is thus found at Brest, that in the syzygies of the summer solstice, the tides of the morning of the first and second day after the syzygy are smaller than those of the evening by about a sixth of a

metre very nearly; they are greater by the same quantity in the syzygies of the winter solstice. In like manner, in the quadrature of the autumnal equinox, the morning tides of the first and second day after the quadrature, surpass those of the evening by about the eighth part of a metre: they are smaller by the same quantity, in the quadratures of the vernal equinox.

Such are, in general, the phenomena which the heights of the tides present in our ports; their intervals furnish other phenomena, which we now proceed to develope.

When the high tide happens at Brest at the moment of the syzygy, it follows the instant of midnight, or that of the true mid day by $0^d,1780$, according as it happens in the morning or in the evening: this interval, which is very different in harbours extremely near to each other, is termed the hour of port, because it determines the hours of the tides relative to the phases of the moon. The high tide which takes place at Brest at the moment of the quadrature, follows the instant of midnight, or of mid day, by 0,358.

The tide which is near to the syzygy, advances or retards $270''$ for each hour by which it precedes or follows the syzygy; the tide which is near to the quadrature, advances or retards $502''$ for each hour it precedes or follows the quadrature.

The hours of the high water in the syzygies and in the quadrature, vary with the distances of the sun and of the moon from the earth, and

principally with the distance of the moon. In the syzygies each minute of increase or diminution in the apparent semidiameter of the moon, advances or retards the hour of high water by 354″. This phenomenon obtains equally in the quadratures, but it is there three times less.

In like manner the declinations of the sun and of the moon influence the hours of high water in the syzygies and in the quadratures. In the solstitial syzygies, the hour of high water advances by about two minutes, and it is retarded by the same quantity in the equinoctical syzygies; on the contrary, in the equinoctial quadratures, the hour of high water advances by about eight minutes, and it is retarded by the same quantity in the solstitial quadratures.

We have seen that the retardation of the tides from one day to another is about 0,03505, in its mean state; so that if the tide happens at 0,1 after the true midnight, it will arrive on the morning after but one at 0^d,13505. But this retardation varies with the phases of the moon. It is the least possible near the syzygies, when the total tides are at their *maximum*, and then it is only 0^d,02723. When the tides are at their *minimum* or near to the quadratures, it is the greatest possible, and amounts to 0^d,05207. Thus, the difference of the hours of the corresponding high water, at the moments of the syzygy and of the quadrature, and which by what precedes is 0^d,20642, increases, for the tides which follow in the same manner these two phases, and becomes

very nearly equal to a quarter of a day, relatively to the *maximum* or the *minimum* of the tides.

The variations of the distances of the sun and of the moon from the earth, and principally those of the moon, influence the retardation of the tides from one day to another. Each minute of increase or of diminution of the apparent semi-diameter of the moon, increases or diminishes this retardation by 258″ towards the syzygies. This phenomenon obtains equally in the quadratures, but it is then three times less.

The daily retardation of the tides varies also with the declination of the two stars. In the solstitial syzygies it is about one minute greater than in its mean state ; it is smaller by the same quantity in the equinoxes. On the contrary, in the equinoctial quadratures it surpasses its mean magnitude by about four minutes ; it is less by the same quantity in the solstitial quadratures.

The results which have been just detailed, were deduced from a series of observations made at Brest since the year 1807, up to the present day. It was interesting to compare them with similar results which have been deduced from observations made at the commencement of the last century. I have found that all the results accord with each other very nearly, their small differences being comprized within the limits to which the errors of observations are liable. Thus, after the interval of a century, Nature has been found agreeing with herself.

Hence it appears that the inequalities of the

heights and of the intervals of the tides have very different periods, the one are equal to half a day and to an entire day, others to half a month, to a month, to half a year, and of a year; and finally others are the same as those of the revolutions of the nodes and of the perigee of the lunar orbit, the position of which influences the height of the tides by the effect of the declinations of the moon, and of its distances from the sun. These phenomena obtain indifferently in all harbours and on the shores of the sea, but local circumstances, without making any change in the laws of the tides, have a considerable influence on their height and the hour of high water for a given port.

CHAP. XVI.

Of the terrestrial atmosphere, and of astronomical refractions.

A RARE elastic and transparent fluid envelopes the earth, and extends to a considerable height. It gravitates (*a*) like all other bodies, and its weight balances that of the mercury in the barometer. At the parallel of fifty degrees, the temperature being supposed to be that of melting ice, and at the mean height of the barometer at the level of the sea, which height may be supposed to be $0^m,76$, the weight of the air is to that of an equal volume of mercury, in the ratio of unity to 10477,9; (*b*) hence it follows that if it be then elevated, by 10^m4779, the height of the barometer will be depressed very nearly one millimetre, and that if the density of the atmosphere was uniform throughout its entire extent, its height would be 7963 metres. But the air is compressible, and if its temperature be supposed constant, its density, according to a general law for gases and fluids reduced to vapours, is proportional to the weight which compresses it, and consequently to the height of the barometer. Its inferior strata being compressed by the superior ones, are consequently more dense than the latter, which become

rarer according as we ascend above the earth's surface. The height of these strata being supposed to increase in arithmetic progression, their density would diminish in geometric progression, provided that the temperature of these strata was the same. In order to understand this, suppose a vertical canal to traverse two atmospherical strata indefinitely near to each. The part of the more elevated stratum through which the canal passes, will be less compressed than the corresponding part of the lower stratum, by a quantity equal to the weight of a small column of air intercepted between these two parts. The temperature being supposed to be the same, the difference of compression of these two strata, is proportional to the difference of their densities; therefore this last difference is proportional to the weight of the small column, and consequently to the product of its density by its length, at least if we abstract from the variation of gravity according as we ascend. The two strata being supposed indefinitely near to each other, the density of the column may be supposed the same as that of the inferior stratum; hence the differential variation of this last density is proportional to the product of this density by the variation of the vertical height; consequently if this height varies by equal quantities, the ratio of the differential (c) of the density to the density itself will be constant; which is the characteristic property of a decreasing geometric progression, all the terms of which are indefinitely near to each other.

Hence it follows that the heights of the strata increasing in arithmetical progression, their densities diminish in geometric progression, and their logarithms, whether hyperbolical or naperian, will decrease in arithmetic progression.

These data have been advantageously applied to the measurement of heights by means of the barometer. The temperature of the atmosphere being supposed to be constant throughout its entire extent, the difference of the heights of the two stations will be obtained by multiplying, by a constant cœfficient, the difference of the logarithms of the observed heights of the barometer at each station. One sole observation is sufficient to determine this coefficient. Thus we have seen that at zero of temperature, the height of the barometer being $0^m,76000$ at the inferior station, and $0^m,75999$ at the superior station, this last station was elevated $0^m,104779$ above the first; consequently the constant coefficient was equal to this quantity divided by the difference of the tabular logarthims of the numbers $0^m,76000$, $0^m,75999$, which renders this coefficient equal to 18336^m. But this rule for measuring heights by means of the barometer, requires several modifications, which we proceed to develope.

The temperature (*d*) of the atmosphere is not uniform; it diminishes according as we ascend. The law of this diminution changes every instant; but a mean result between several observations gives sixteen or seventeen degrees for the diminution of the temperature relative to an

height of three thousand metres. Now the air, like all other bodies, expands by heat, and contracts by cold; and it has been found by very accurate experiments, that its volume being represented by unity, at the temperature of zero, it varies like that of all gazes and vapours by 0,00375 for *(f)* each degree of the thermometer; it is therefore necessary to take these variations into account in the computation of heights, for it is evident that in order to produce the same depression in the barometer, it is necessary to ascend so much the higher, as the stratum of air through which we must pass is rarer. But as it is impossible to know accurately the variation of the temperature, the simplest method of proceeding is to suppose this temperature uniform, and a mean between the temperatures of the two stations which are considered. The volume of the column of air comprised between them being increased in the ratio of this mean temperature, the height due to the observed depression of the barometer must be increased in the same ratio, which comes to multiplying the cœfficient 18336^{m}, by unity plus the fraction 0,00375, taken as often as there are degrees in the mean *(g)* temperature. As the aqueous vapours which are diffused through the atmosphere are less dense than the air at the same pressure and temperature, they diminish the density of the atmosphere, and every thing else being the same, they are more abundant when the heat is greater; this effect will be partly taken into account, by increasing a little the number 0,00375,

which expresses the dilation of the air for each degree of the thermometer. It has been ascertained that the observations are sufficiently well satisfied by making this fraction equal to 0,004; we may therefore make use of this last number, at least until by a long series ofobservations on the hygrometer, we are enabled to introduce this instrument in the measurement of heights by the means of the barometer.

Hitherto the force of gravity has been supposed to be constant, but it has been already observed that it is less according as we ascend in the atmosphere; this circumstance also contributes to increase the height due to the depression of the barometer, consequently this diminution of gravity will be taken into account, if the constant factor be increased by a small quantity. From a comparison of a great number of observations of the barometer, made at the base and at the summit of several mountains, the heights of which were previously ascertained by trigonometrical means, Raymond has determined this factor to be equal to 18393^{m}. But if the (h) diminution of gravity be taken into account, a comparison of the same data would only give this factor equal to 18336^{m}. This last factor gives 10477,9 for the ratio of the weight of mercury to that of an equal volume of air at the parallel of fifty degrees; the temperature as indicated by the barometer being zero, and the height of the mercury in the barometer being $0^{m},76$. Biot and Arrago having carefully weighed known measures of mercury and of

air, found this ratio to be 10466,6 reduced to the same parallel. But they made use of very dry air, while that of the atmosphere is always mixed with a greater or less quantity of aqueous vapour, the actual quantity of which is determined by means of the hygrometer: this vapour is lighter than the air in the ratio of ten to seventeen very nearly; consequently direct experiment ought to assign a less specific gravity to mercury than that determined by barometrical observations. These experiments reduce the factor 18336^{m} to 18316^{m}. In order that it should be supposed equal to the number 18393^{m}, which is given by observations of the barometer, when the variation of gravity is not taken into account, we should assign to the mean humidity of the atmosphere a value much too great; thus the diminution of gravity is even sensible in barometrical observations. The factor 18393 corrects very nearly the effect of this diminution, but another variation of gravity, namely, that which depends on the latitude, ought also to influence this factor. We have determined it for a parallel of which the latitude may without sensible error be supposed equal to 50°: it should therefore be increased at the equator where the gravity is less *(i)* than at this latitude. In fact, it is evident that it should be elevated more, in order to pass from a given pressure of the atmosphere to a pressure which is smaller by a determined quantity; because in this interval the weight of the air is less, the cœfficient 18393^{m} must therefore vary as the length of the pendulum which vibrates se-

conds, which is greater or less according as the gravity diminishes or increases. It is easy to infer from what has been previously stated relative to the variations of this length, that the product of $26^{m},164$ by the cosine of twice the latitude, must be added to this coefficient.

Finally, a slight correction should be applied to the heights of the barometer, depending on the difference of temperature of the mercury of the barometer at the two stations. In order to determine this difference accurately, a small mercurial thermometer is inchased in the frame of the barometer, so that the temperature of the mercury of these two instruments may be very nearly the same. In the colder station the mercury is denser, and consequently the column of mercury of the barometer is diminished. In order to reduce it to the length which it would *(k)* have, if the temperature was the same as at the warmer station, it should be increased by its 5412^{mth} part, as often as there are degrees of difference between the temperatures of the mercury at the two stations.

Hence the following appears to be the simplest and most exact rule for measuring heights by means of the barometer. First, the height of the barometer in the colder station must be corrected in the manner just specified. Then to the factor 18393^{m}, should be added the product of 26,164 by the cosine of twice the latitude. This factor, thus corrected, should be multiplied by the tabular logarithm of the ratio of the greatest to

the least corrected height of the barometer. Finally, this product must be multiplied by twice the sum of the degrees of the thermometer which indicates the temperature of the air at each station, and this product, divided by one thousand, should be added to the preceding; the sum will give very nearly the elevation of the superior station above the inferior, especially if the observations of the barometer are made at the most favourable time of the day, which appears to be at noon.

The air is invisible in small masses, but the rays of light reflected by all the strata of the terrestrial atmosphere, produce a sensible impression. They (*l*) give it a blue shade which diffuses a tint of the same colour over all objects perceived at a distance, and which forms the celestial azure. This blue vault, to which the stars appear to be attached, is therefore very near to us: it is only the terrestrial atmosphere, beyond which these bodies are placed at (*m*) immense distances. The solar rays, which its particles reflect to us in abundance before the rising and after the setting of the sun, produce the dawn and twilight, which, extending to more than twenty degrees of distance from this star, proves that the extreme particles of the atmosphere are elevated at least sixty thousand metres. If the eye could distinguish and refer to their true place, the points of the exterior surface of the atmosphere, we should see the heavens like the segment of a sphere formed by the portion of the surface which would be cut off

by a plane tangent to the earth; and as the height of the atmosphere is very small relatively to the radius of the earth, the sky would appear to us under the form of a flattened vault. But although the limits of the atmosphere cannot be distinguished, yet as the rays which it transmits come from a greater depth at the horizon than at the zenith, we ought to consider it as more extended in the first direction. To this cause must be also combined the interposition of objects at the horizon, which contributes to increase the apparent distance of that part of the sky we refer to it; the sky therefore should appear to us very much flattened, like a small portion of a sphere. A star, elevated twenty-six degrees above the horizon, appears to divide into two equal parts the length of the curve which the section of the surface of the sky by a vertical plane forms from the horizon to the zenith; hence it follows that if this curve be an arc of a circle, the horizontal radius of the apparent celestial vault (*n*) is to its vertical radius very nearly as three fourths is to unity; but this ratio varies with the causes of the illusion. As the apparent magnitudes of the sun and of the moon are proportional to the angles under which they are seen, and to the apparent distance of the point of the sky to which they are referred, they *appear* greater at the horizon than at the zenith, although they subtend a smaller angle.

The rays of light do not move in a right line through the atmosphere, they are continually

inflected towards the earth. As an observer beholds objects in the direction of the tangent to the curve which they describe, he sees them more elevated than they really are, so that the stars appear above the horizon when they are depressed below it. By this means the atmosphere, by inflecting the rays of the sun, lengthens the time during which he appears to us, and thus prolongs the duration of the day, which is further increased by the morning dawn and twilight. It is extremely important to astronomers, to know the laws and quantity of the refraction of light in our atmosphere, in order to be able to determine the position of the stars. But before we present the result of their researches on this subject, we shall briefly explain the principal properties of light.

A ray of light, in passing from one transparent medium into another, approaches to, or recedes from the perpendicular to the surface which separates them, in such a manner, that the sines of the two angles which its directions make with this perpendicular, the one before and the other after its entrance into the new medium, are in a constant (*o*) ratio, whatever be the magnitude of these angles. But light, when refracted, presents a remarkable phenomenon, which has led to the discovery of its nature. A ray of solar light received into a dark chamber forms, after its passage through a prism, an oblong image variously coloured; this ray is a pencil of an infinite number

of rays of different colours, which are separated by the prism in consequence of their different refrangibility. The most refrangible ray is the violet, then the indigo, the blue, the green, the yellow, the orange and the red. But though we only distinguish seven species of rays, the continuity of the image proves that there exists an infinite variety of shades, which approach each other by insensible gradations of colours and refrangibility. All these rays being collected by means of a lens, reproduce the white light of the sun, which is therefore only a mixture of all the homogeneous or simple colours in determined proportions.

When a ray of an homogeneous colour is perfectly separated from the others, it does not change either its refrangibility or colour, whatever reflexions or refractions it may undergo; therefore its colour is inherent in its nature, and not a modification which light receives in the media which it traverses. However, a similitude of colour does not prove a similitude of light. If several of the differently coloured rays of the solar image, decomposed by the prism, be mixed together, a colour perfectly similar to one of the simple colours of this image will be formed; thus the mixture of the homogeneous red and yellow produces an orange similar in appearance to the homogeneous orange. But by refracting the rays of this mixture by a second prism, the component colours can be separated and made to reappear, while the rays of the homogeneous orange

remain unaltered. The rays of light are reflected when they fall on a mirror, making, with the perpendicular to its surface, the angles of reflexion equal to the angles of incidence. The refractions and reflections which rays of light undergo in drops of rain, produce the rainbow, the explanation of which, founded on a rigorous computation which satisfies all the phenomena, is one of the most beautiful results of natural philosophy.

Most bodies decompose the light which they receive; they absorb one part and reflect the other in every direction; they appear blue, red, green, &c. according to the colour of the rays which they reflect. Thus the white light of the sun diffusing itself over all (*p*) natural objects, decomposes and reflects to our eyes an infinite variety of colours.

After this short digression, we return to astronomical refractions. From very accurate experiments it has been ascertained that the refraction of the air is almost independent of its temperature, and proportional to its density. In passing from a vacuo into air, of which the temperature is equal to that of melting ice, and under a pressure measured by a height of the barometer equal to 76 centimetres, a ray of light is so refracted that the sine of incidence is to the sine of refraction as 1,0002943321 to 1. Therefore in order to determine the rout of a ray of light through the atmosphere, it is sufficient to know the law of the density of its strata; but this law, which depends on their temperature, is very complicated, and varies for every instant of the

day. We have seen already, that when the atmosphere is throughout at the temperature of zero, the density of the strata (q) diminish in geometric progression; and it has been found by analysis, that the height of the barometer being $0^m,76$, the refraction is then $7391''$ at the horizon. It would be but $5630''$ if the density of the strata diminished in arithmetic progression, and vanished at the surface. The horizontal refraction which is observed is about $6500''$, a mean between these limits. Consequently the law of the diminution of the density of the atmospherical strata is very nearly a mean between these two progressions. By adopting an hypothesis which participates of the two, we are enabled to represent at once all the observations of the barometer and thermometer, according as we ascend in the atmosphere, and also the astronomical refractions, without having recourse, as some natural philosophers have done, to a particular fluid, which, being combined with the atmospheric air, refracts the light.

When the apparent altitude of the stars above the horizon exceeds eleven degrees, their refraction depends only sensibly on the state of the barometer and thermometer at the place of the observer, and it is very nearly proportional to the tangent of the apparent zenith distance of the star, diminished by three times and one fourth of the refraction corresponding to (r) this distance at the temperature of melting ice, the height of the barometer being $0^m,76$. It follows from the pre-

ceding data that at this temperature, and when the height of the barometer is seventy-six centimetres, the coefficient, which multiplied by this tangent gives the astronomical refraction, is 187″,24; and what is very remarkable, a comparison of a great number of astronomical observations gives the same result, which must therefore be supposed extremely accurate; but it varies with the density of the air. Each degree of the thermometer increases by 0,00375 the volume of this fluid, its unity being assumed at zero of the temperature, it is therefore necessary to divide the coefficient 187″,24 by unity, plus the product of 0,00375 into the number of degrees indicated by the thermometer; moreover, the density of the air is, every thing else being the same, proportional to the height of the barometer; it is therefore necessary to multiply the preceding coefficient by the ratio of this height to $0^{m},76$, the column of mercury being reduced to the zero of temperature. By means of these data a very exact table of refractions may be constructed, from eleven degrees of apparent altitude to the zenith, in which interval almost all astronomical observations are made.

This table will be independent of all hypotheses relative to the diminution of the density of the atmospherical strata, and might as well be applied at the summit of the highest mountains as at the level of the sea. But as the gravity varies with the elevation and latitude, it is evident, that as at the same temperature, equal heights of the

barometer do not indicate an equal density in the air, this density must be less in those places where the gravity is less. Thus the coefficient 187″,24, determined for the parallel of 50°, must *at the surface of the earth* (*s*) vary as the weight, it is therefore necessary to subtract from it the product of 0″,53 by the cosine of twice the latitude.

The table of which we have been speaking, supposes that the constitution of the atmosphere is every where and always the same, which has been proved by direct experiment. It is now ascertained that our atmosphere is not an homogeneous substance, and that in every hundred parts, it contains 79 parts of azotic gas, and 21 parts of oxygen gas, a gas remarkably respirable, which is indispensably necessary for the combustion of bodies (*t*) and the respiration of animals, which is in fact but a slow combustion, the principle source of animal heat; three or four parts of cartonic acid air are diffused in a thousand parts of atmospheric air. This air, taken at all seasons, in the most remote climates, on the summits of the highest mountains, and even at greater heights, has been most carefully analyzed, and it has always been found to contain the same proportions of azotic and oxygen gas. A slight envelope filled with hydrogen gas, the rarest of all elastic fluids, ascends with the bodies which are attached to it, untill it meets with a stratum of the atmosphere sufficiently rare for it to remain (*u*) in equilibrium. By this means, for the fortunate discovery of which we are indebted to the French philosophers,

man has extended his power and sphere of action; he may launch into the air, traverse the clouds, and interrogate nature in the elevated regions of the atmosphere formerly inaccessible. The ascent from which the greatest advantages have been derived to the sciences, was that of Gay-Lussac, who ascended to a height of seven thousand and sixteen metres above the level of the sea, the greatest height to which an aeronaut has hitherto attained, and which is higher than the top of Chimboraco, one of the highest known mountains, by about five hundred metres. At this elevation, he measured the intensity of the magnetic force, the inclination of the magnetized needle, which he found to be the same as at the surface of the earth. At the instant of his departure from Paris, near to ten o'clock A. M. the height of the barometer was $0^{m},7652$, the thermometer indicated $30^{\circ},7$, and a hygrometer made of hair, 60°. Five hours after, at the greatest height to which he ascended, the same instruments indicated respectively $0^{m},3288$;—$9^{\circ},5$ and 33°. A balloon having been filled with the air of these elevated strata, and its contents being then carefully analyzed, the contents were found to be precisely the same as those of the lowest strata of the atmosphere.

It is not more than half a century since astronomers introduced the consideration of the heights of the barometer and thermometer, into the tables of refractions. The great precision which is now required in instruments and astronomical observations, makes it a matter of importance to as-

certain whether the humidity of the atmosphere has any influence on the refracting force, and consequently to know whether it is necessary to take into account the indications of the hygrometer.

In order to supply the defect of direct experiment on this subject, let us suppose (v) that the action of water and vapour on light are proportional to their densities, which hypothesis is extremely probable from the circumstance, that changes in the constitution of bodies much more essential, than the reduction of liquids into vapours do not alter in a sensible degree the relation of their action on light, to their density. In this hypothesis, the refracting power of the aqueous vapour may be inferred from the refraction which a ray of light experiences in passing from air into water, which refraction has been exactly measured. It has been thus ascertained that this refracting power surpasses that of air reduced to the same density as the vapour; but at equal pressures, the density of the air surpasses that of vapour in very nearly the same ratio; hence it follows that the refraction due to the aqueous vapour diffused through the atmosphere, is very nearly the same as that of the air of which it occupies the place, and that consequently the effect of the humidity of the air on the refraction is insensible. Biot has confirmed this result by direct experiments, which shew moreover that the temperature does not influence the refraction, except so far as it produces a change in the density of the air. Finally, Arrogo ascertained, by

an ingenious and accurate method, that the influence of the humidity of the air on its refraction is altogether insensible.

It is supposed in the preceding theory that the atmosphere is perfectly calm, so that the density is every where the same at equal heights above the level of the sea. But this hypothesis is affected by winds and inequalities of temperature, which must influence in a very sensible manner the astronomical refractions. However perfect astronomical instruments may be rendered, the effect of these perturbating causes, if it is considerable, will be always an obstacle to the extreme accuracy of observations, which should be multiplied considerably in order to annihilate them. Fortunately we are assured that this effect can never exceed a small number of seconds.

The atmosphere weakens the light of the stars, especially near the horizon, where their rays transverse through a greater extent of it. It follows, from the experiments of Bouguer, that when the height of the barometer is seventy-six centimetres, if the intensity of the light of a star at its entrance into the atmosphere be represented by unity, its intensity when it arrives at the observer, the star being supposed to be in the zenith, will be reduced 0,8123. The height of the homogeneous atmosphere, of which the temperature was zero, would in this case be 7945^{m}. (x) Now it is natural to suppose that the extinction of a ray of light which traverses the atmosphere, is the same as in this hypothesis, since it meets with

the same number of aerial particles, consequently a stratum of air of the preceding density, and of which the thickness was 7945^m, would reduce the force of light to $0^m,8123$. It is easy to determine from hence the diminution of light in a stratum of air of the same density, and of any given thickness; for it is evident that if the density of light is reduced to a fourth in traversing a given thickness, an equal thickness will reduce this fourth to a sixteenth of its primitive value; hence it appears that while the thickness increases in arithmetical progression, the intensity of light decreases in geometrical progression; consequently its logarithms are proportional to the thickness. Thus in order to obtain the tabular logarithm of the intensity of light after it has traversed any given thickness, it is necessary to multiply $-0,0902835$ (which is the tabular logarithm of $0,8123$) by the ratio of this thickness to 7945^m; and if the density of the air is greater or less than the preceding, it is necessary to diminish this logarithm in the same ratio.

In order to determine the diminution of the light of the stars with respect to their apparent altitude, we may suppose the luminous ray to move in a canal, the air in this canal being reduced to the preceding density. The length of the column of air thus reduced, will determine the extinction of the light of the star which is considered; now we may suppose that from twelve degrees of apparent elevation to the zenith, the path of the light of the stars is rectilineal, and we

can, in this interval, consider the atmospherical strata as planes parallel to each other; then the thickness of each stratum in the direction of the ray of light, is to its thickness in a vertical direction, as the secant of the apparent distance of the star from the zenith, (y) is to radius. Therefore if this secant be multiplied by —0,0902835, and by the ratio of the height of the barometer to $0^m,76$, and if this product be then divided by unity plus 0,00375 multiplied by the number of degrees in the thermometer, we shall have the logarithm of the intensity of light of the star. This rule, which is extremely simple, will determine the extinction of the light of the stars on the summit of mountains and at the level of the sea, and may be usefully applied, both in correcting the observations of the eclipses of Jupiter's satellites, and also in estimating the intensity of solar light in the focus of burning glasses. It ought however to be observed, that vapours floating in the air influence considerably the extinction of light. The serenity of the sky and the rarity of the air make the light of the stars more brilliant on the tops of elevated mountains, and if we could transport our great instruments to the summit of the Cordilieres, there is no doubt but that we should observe several celestial phenomena, which a thicker and less transparent atmosphere renders invisible in our climates.

The intensity of the light of the stars at small altitudes like to their refraction, depends on the density of the elevated strata of the atmosphere.

If the temperature was every where the same, the logarithms of the intensities of light would be proportional to the astronomical refractions (z) divided by the cosines of the apparent heights; and then this intensity at the horizon would be reduced to about the four thousandth part of its primitive value; it is on this account that the sun, whose splendour at noon is too dazling to be borne, can be contemplated without pain at the horizon.

We can by means of these data determine the influence of our atmosphere in eclipses. As it refracts the rays of the sun which traverse it, it inflects them into the cone of the terrestrial shadow, and as the horizontal refractions surpasses the semi sum of the parallaxes of the sun and moon, the centre of the lunar disk supposed to exist on the axis of the cone, receives from the upper and lower limbs of the earth the rays which issue from the same point of the sun's surface; this centre would be therefore more illuminated than in full moon, if the atmosphere did not in a great measure extinguish the light which reaches it. If the light of this point at full moon be taken for unity, it is found by applying the analysis to the preceding data, that the light is 0,02 in the central apogean eclipses, and only 0,0036, or about six times less, in the central perigean eclipses. If it then happens by an extraordinary concurrence of circumstances, that the vapours absorb a considerable part of this feeble light, when it traverses the atmosphere (a) in passing from the sun to the

moon, this last star will be altogether invisible. The history of astronomy furnishes us with examples, of rare occurrence indeed, of the total disappearance of the moon in eclipses. The red colour of the sun and moon at the horizon shews that the atmosphere gives a free passage to the rays of this colour, which, on this account, is that of the moon when eclipsed.

In eclipses of the sun, the obscurity which they produce is diminished by the light reflected by the atmosphere. Suppose in fact, the spectator to be placed in the equator, and that the centres of the sun and moon are in his zenith. If the moon was in perigee, the sun would be in the direction of the apogee; in this case the obscurity would be very nearly the most profound, and its duration would be about five minutes and a half. The diameter of the shadow projected on the earth will be the twenty-two thousandth part of that of the earth, and six times and a half less than the diameter of the section of the atmosphere by the plane of the horizon, at least if we suppose the height of the atmosphere equal to a hundreth part of the earth's radius, which is the height inferred from the duration of twilight; and it is very probable that the atmosphere reflects sensible rays from still greater heights. It appears therefore, that in eclipses, the sun illuminates the greater part of the atmosphere which is above the horizon. But it is only illuminated by a portion of the sun's disk, which increases according as the atmospherical molecules are more distant from the ze-

nith; in this case the solar rays traverse a greater extent of the atmosphere, in passing from the sun to these molecules, and after this in returning by reflexion to the observer, they are sufficiently diminished in intensity to enable us to perceive stars of the first and second magnitude. Their tint, participating of the blue colour of the sky, and of the red colour of twilight, diffuses over all objects a sombre colour, which combined with the sudden disappearance of the sun, fills all animals with terror.

BOOK THE SECOND.

OF THE REAL MOTIONS OF THE HEAVENLY BODIES.

Provehimur portu, terræ urbesque recedunt.

A COMPARISON of the principal appearances of the heavenly bodies, of which the exposition has been given in the preceding book, has led us to make the planets move round the sun, which in its revolution round the earth, carries along with it the foci of their orbits. But the appearances would be precisely the same, if the earth was transported, like the other planets, about the sun: then this star would be the centre of the planetory motions in place of the earth. It is consequently of the greatest importance to the progress of astronomy, to ascertain which of these two cases obtains in nature. We therefore proceed, under the guidance of induction and analogy, to determine, by a comparison of phenomena, the real motions which produce them, and thence to ascend to the laws of these motions.

CHAP. I.

Of the motion of rotation of the earth.

When we reflect on the diurnal motion to which all the heavenly bodies are subject, we evidently recognize the existence of one general cause which moves, or appears to move them about the axis of the world. If it be considered that these bodies are detached from each other, and placed at very different distances from the earth; that the sun and stars are much more removed than the moon, and that the variations of the apparent diameters of the planets indicate great changes in their distances; lastly, that the comets traverse the heavens freely in all possible directions, it will be extremely difficult to conceive that one and the same cause impresses on all these bodies, a common motion of rotation: but since the heavenly bodies present the same appearances to us, whether the firmament carries them about the earth, supposed immoveable, or whether the earth itself revolves in a contrary direction, it seems much more natural to admit this latter motion, and to regard that of the heavens as only apparent.

The earth is a globe, of which the radius is only about four thousand metres; the magnitude of the sun is, as we have seen, incomparably greater. If its centre coincided with that of the earth, its volume would embrace the orb of

the moon, and extend as far again; from which we may form some judgment of its immense magnitude, besides its distance from us is about twenty-three thousand times the semidiameter of the earth. Is it not infinitely more probable to suppose that the globe which we inhabit revolves on an axis, than to imagine that a body so considerable and distant as the sun, should revolve with the rapid motion which it should have in order that it might revolve in a day, about the earth? What immense force must it not then require to keep it in its orbit, and to counterbalance its centrifugal force. Each of the stars presents similar difficulties, all of which are removed by supposing the earth to revolve on its axis.

It has been already observed, that the pole of the equator appears to move slowly about that of the ecliptic, from whence results the precession of the equinoxes. If the earth be immoveable, the pole of the equator will be equally so, because it always corresponds to the same point of the earth's surface; consequently the celestial sphere moves round the poles of the ecliptic, and in this motion, it carries along with it all the heavenly bodies. Thus the entire system, composed of so many bodies, differing from each other, in their magnitudes, their motions, and their distances, would be again subject to a general motion, which disappears, and is reduced to a mere appearance, if the axis of the earth be supposed to move round the poles of the ecliptic.

Carried along with a motion in which all the

surrounding bodies participate, we are like to a mariner borne by the winds over the seas. He supposes himself to be at rest, and the shore, the hills, and all the objects situated beyond the vessel, appear to him to move. But on comparing the extent of the shore, the plains, and the height of the mountains, with the smallness of his vessel, he is enabled to distinguish the apparent motion of these objects from a real motion to which he himself is subject. The innumerable stars distributed through the celestial regions are, relatively to the earth, what the shore and mountains are with respect to the navigator; and the very same reasons which convince the navigator of the reality of his own motion, evince to us that of the earth.

These arguments are likewise confirmed by analogy. A motion of rotation has been observed in almost all the planets, the direction of which is from west to east, similar to that which the diurnal motion of the heavens seems to indicate in the earth. Jupiter, whose magnitude is considerably greater than that of the earth, revolves on an axis, in less than half a day. An observer on his surface would suppose that the heavens revolved round him in that time; yet that motion would be only apparent. Is it not therefore reasonable to suppose that it is the same with that which we observe on the earth? What confirms, in a very striking manner, this analogy is, that the earth and Jupiter are flattened at the poles. In fact, we may conceive that the centri-

fugal force which tends to make every particle of a body recede from its axis of rotation, should flatten it at the poles and elevate it at the equator. This force should likewise diminish that of gravity at the equator; and that this diminution does actually take place, is proved by experiments which have been made on the lengths of pendulums. Every thing therefore leads us to conclude that the earth has really a motion of rotation, and that the diurnal motion of the heavens is merely an illusion which is produced by it; an illusion similar to that which represents the heavens as a blue vault to which all the stars are attached, and the earth as a plain on which it rests. Thus astronomy has surmounted the illusions of the senses, but it was not till after they were dissipated by a great number of observations and computations, that man at last recognized the motion o. the globe which he inhabits, and its true position in the universe.

CHAP. II.

Of the motion of the earth about the sun.

SINCE it appears from the preceding chapter that the diurnal revolution of the heavens is an illusion produced by the rotation of the earth, it is natural to think that the annual revolution of the sun, carrying with it all the planets, is also an illusion arising from the motion of translation of the earth about the sun. The following considerations remove all doubt on this subject.

The masses of the sun and of several of the planets are considerably greater than that of the earth; it is therefore much more simple to make the latter to revolve about the sun, than to put the whole solar system in motion about the earth. What a complication in the heavenly motions would the immobility of the earth suppose? What a rapid motion must be assigned to Jupiter, to Saturn, (which is nearly ten times farther from the sun than we are) and to Uranus (which is still more remote,) to make them revolve about us every year, while they move about the sun.? This complication and this rapidity disappear entirely by supposing the earth to revolve about the sun, which motion is conformable to a general

law, according to which the small celestial bodies revolve about the larger ones, which are situated in their vicinity.

The analogy of the earth with the planets, confirms the supposition of the earth's motion; like Jupiter it revolves on its axis, and is accompanied by a satellite. An observer at the surface of Jupiter, would suppose that the whole solar system revolved about him, and the magnitude of the planet would render this illusion less improbable than for the earth. Is it not therefore natural to suppose that the motion of the solar system round us, is likewise only an illusion?

Let us transport ourselves in imagination to the surface of the sun, and from thence let the earth and planets be contemplated. All these bodies would appear to move from west to east; this identity in the direction is an evident proof of the motion of the earth; but what evinces it to a demonstration, is the law which exists between the times of the revolutions of the planets, and their distances from the sun. The angular motions are slower for those bodies which are more removed from the sun, and the following remarkable relation has been observed to exist between the times and the distances, namely, that the squares of the times are as the cubes of their mean distances from this star. According to this remarkable law, the duration of the revolution of the earth, supposed to move above the sun, should be exactly a sidereal year. Is not this an incontestable proof that the earth moves like the other planets, and that

it is subject to the same laws? Besides, would it not be absurd to suppose that the terrestrial globe, which is hardly visible at the sun, is immoveable amidst the other planets which are revolving about this star, which would itself be carried along with them about the earth? Ought not the force which balances the contrifugal force, and retains the planets in their respective orbits, act also on the earth, and must not the earth oppose to this action the same centrifugal force? Thus the consideration of the planetary motions, as seen from the sun, removes all doubt of the real motion of the earth. But an observer placed on this body, has besides a sensible proof of this motion, in the phenomena of the aberation which is a necessary consequence of it, as we shall now explain.

About the close of the 17th century, Roemer observed that the eclipses of Jupiter's satellites happened sooner than the computed time near the oppositions of this planet with the sun, and that they occurred later towards the conjunctions; this led him to suspect that the light was not transmitted instantaneously *(a)* from these stars to the earth, and that it employed a sensible interval in passing over the diameter of the orbit of the sun. In fact, Jupiter being in the oppositions, nearer to us than in the conjunctions, by a quantity equal to this diameter, the eclipses must happen sooner in the first case than in the second, by the time which the light takes to traverse the solar orbit. The law of the retardation observed in these eclipses, corresponds so exactly to this

hypothesis, that it is impossible to refuse our assent to it. It follows therefore that light employs 571″, in coming from the sun to the earth.

Now an observer at rest would see the stars in the direction of their rays, but this is not the case, on the hypothesis that he moves along with the earth. In order to reduce this case to that of a spectator at rest, it would be sufficient to transfer in a contrary direction both to the stars, to their light, and to the observer himself, the motion with which he is actuated, which does not make any change in the apparent position of the stars; for it is a general law of optics, that if a common motion be impressed on all the bodies of a system, there will not result any change in their apparent situation. Suppose then that at the instant a ray of light penetrates the atmosphere, a motion equal and contrary to that of the observer be impressed on the air and the earth; and let us consider what effects this motion ought to produce in the apparent position of the star from which the ray emanates. We may leave out of the question the consideration of the motion of rotation of the earth, which is about sixty times less at the equator itself, than that of the earth about the sun, and we may also, without sensible error, suppose that all the rays of light which each point of the star's disk transmits to us, are parallel to each other, and to the rays which would pass from the centre of the star to that of the earth, on the hypothesis that it was transparent. Thus the phenomena which these stars

would present to a spectator situated at the centre of the earth, and which depend solely on the motion of light combined with that of the earth, are very nearly the same for each observer on its surface. Finally, we may neglect the small excentricity of the earth's orbit. This being premised, in the interval of 571″, that light takes to traverse the radius of the earth's orbit, the earth describes a small arc of this orbit equal to 62″5; now it follows from the composition of motions, that if through the centre of the star a small circle parallel to the ecliptic be described, the diameter of which subtends in the heavens an arc of 125″, the direction of the motion of light, when compounded with the motion of the earth applied in a contrary direction, meets this circumference at the point where it is intersected by a plane drawn through the centre of the star and of the earth tangentially to the terrestrial orbit, the star must therefore appear to move in this circumference, and to describe it in (*c*) a year, in such a manner that it is always less advanced by one hundred degrees, than the sun in his apparent orbit.

This is precisely the phenomenon which has been explained in the eleventh chapter of the first book, from the observations of Bradley, to whom we are indebted for its discovery and that of its cause. The true place of the stars is the centre of the small circumference which they appear to describe; their annual motion is only an illusion produced by the combination of the motion of

light with that of the earth. From its evident relations with the position of the sun, it might be justly supposed that it was only apparent; but the preceding explanation proves it to a demonstration. It also furnishes a sensible proof of the motion of the earth about the sun, in the same manner as the increase of degrees and of the force of gravity from the equator to the poles, proves the rotation of the earth on its axis.

The aberation of light affects the positions of the sun, the planets, the satellites, and the comets, but in a different manner from the fixed stars, in consequence of their respective motions. In order to divest them of this, and to obtain the true position of the stars, we should impress at each instant, on all these bodies, a motion equal and contrary to that of the earth, which by this means becomes immoveable, and which, as has been already observed, neither changes their respective positions, nor their appearances. It is evident then that the star, at the moment that it is observed, has not the direction of the rays of light which strike our eye; it deviates from it in consequence of its real motion combined *(d)* with that of the earth, which we suppose to be impressed on it in a contrary direction. The combination of these two motions, when observed from the earth, produces the apparent, or as it is termed *the geocentrick motion*. Therefore the true position of the star will be obtained by adding to its observed geocentrick longitude or latitude, its geocentrick motion in longitude and in latitude, in the

interval of time which light takes to come from the star to the earth. Thus the centre of the sun always appears to us less advanced in its orbit by $62'',5$, than if the light was transmitted to us instantaneously.

The aberration changes the apparent relations of the celestial phenomena with respect both to their situation and duration. At the moment we see them they no longer exist. The satellites of Jupiter have ceased to be eclipsed twenty-five or thirty minutes, when we observe the termination of the eclipse; and the variations of the changeable stars precede by several years, the instant at which they are observed. But the cause of all these illusions being well understood, we can always refer the phenomena of the solar system, to their true place and exact epoch.

The consideration of the celestial motions leads us, then, to displace the earth from the centre of the world, where we had placed it, deceived by appearances, and by the natural propensity of man to regard himself as the principal object of nature. The globe which he inhabits is a planet in motion on itself and about the sun. When it is considered in this point of view, all the phenomena are explained in the simplest manner; the laws of the celestial motions are rendered uniform, and the analogies are all observed. Thus, like Jupiter, Saturn and Uranus, the earth is accompanied by a satellite; it revolves on an axis like Venus, Mars, Jupiter and Saturn, and probably all the other planets; like them it borrows

its light from the sun, and revolves about him in the same direction, and according to the same laws. Finally, the hypothesis of the earth's motion combines in its favour simplicity, analogy, and generally every thing which characterises the true system of nature. We shall see, by following it in all its consequences, that the celestial phenomena are reduced in their minutest details to one sole law, of which they are the necessary developements. The motion of the earth will thus acquire all the certainty of which physical truths are susceptible, and which may result either from the great number and variety of phenomena which it explains, or from the simplicity of the laws on which it is made to depend. No branch of natural science combines in a higher degree these criterea than the theory of the system of the world, founded on the motion of the earth.

This motion enlarges our conceptions of the universe, by furnishing for a measure of the distance of the heavenly bodies, an (*e*) immense base, namely, the diameter of the earth's orbit; by means of this, the dimensions of the planetary orbits have been exactly determined. Thus the motion of the earth, after having, by the illusions of which it was itself the cause, retarded our knowledge of the planetary motions for a great length of time, has at last conducted us to a knowledge of them, and that in a more accurate manner than if we had been placed at the focus of these motions. Nevertheless the annual parallax of the fixed stars, or the angle which the diameter of the

earth's orbit would subtend at this centre, is insensible, and does not amount to 6″, even relatively to those stars which *(f)* from their great brilliancy appear to be nearest to us; they are therefore at least two hundred thousand times farther from us than the sun. Their great brilliancy, at such an immense distance, proves to us that they do not, like the planets and satellites, borrow their light from the sun, but that they shine with their own proper light; so that they may be considered as so many suns distributed in the immensity of space, and similarly to our own, may be the foci of so many planetary systems. It would in fact be sufficient to place ourselves at the nearest of those stars, in order to see the sun as a luminous star, the diameter of which was less than the thirtieth part of a second.

It follows from the immense distances of the stars, that their motions in right ascension and declination are only apparent, and that they are produced by the motion of the earth's axis of rotation. But some stars appear to have motions proper to themselves, and it is probable that all of them are in motion as well as the sun, which carries with it in space the entire system of the planets and comets, in the same manner as each planet carries along with it, its satellites in their motions about the sun.

CHAP. III.

Of the appearances which arise from the motion of the earth.

From the point of view in which a comparison of the celestial phenomena has placed us, let us consider the stars, and shew the perfect identity of their appearances with those which we observe. Whether the heavens revolve about the axis of the world, or the earth revolves on its axis in a contrary direction to the apparent motion of the heavens, supposed to be at rest, it is clear that the appearances of the stars, on either hypothesis will be precisely the same. The only difference will be, that in the first case they will place themselves over the different terrestrial meridians, which, in the second case, will place themselves under these stars.

The motion of the earth being common to all bodies situated on its surface, and also to the fluids which cover it, their relative motions are the same as if it was immoveable. Thus, in a vessel transported with an uniform motion, every body moves as if it was in a state of rest. A projectile thrown directly upwards falls on the same spot from which it was projected; it appears to those in the vessel to describe a vertical line, but

to a spectator on the shore, it will appear to move obliquely to the horizon, and to describe a curve which is sensibly parabolic. However, the real velocity which arises from the rotatory motion of the earth being somewhat less at the bottom than at the top of an elevated tower, if a body be let fall freely from this top, it is evident that in consequence of the excess of its real velocity of rotation above that of the bottom of the tower, it should not fall exactly at the point where the plumb line from the summit of the tower meets the surface of the earth, but a little to the east. In fact, it appears from analysis that its deviation from this point towards the east, is proportional to the sesquiplicate ratio of the height of the tower, and to the cosine of latitude, (*a*) and that at the equator it is but $21^{m},952$ for one hundred metres of height. We may therefore, by means of very accurate experiments on falling bodies, render the rotation of the earth sensible. Those which have been already instituted with this view in Germany and Italy, agree sufficiently well with the preceding results; but these experiments which require the most delicate manipulation, ought to be repeated with still greater precision. The rotation of the earth is principally indicated at its surface, by the effects of the centrifugal force, which flattens the terrestrial spheroid at the poles, and diminishes the gravity (*b*) at the equator, two phenomena, which the measures of the pendulum and of the degrees of the meridian, have made known to us.

In the revolution of the earth about the sun, its centre and all the points of its axis of rotation move with equal and parallel velocities; this axis therefore remains (c) always parallel to itself. If at every instant, a motion equal and contrary to that of the earth's centre be impressed on the heavenly bodies, and also on all the parts of the earth, this centre would remain immoveable, as also its axis of rotation; but this impressed motion does not change at all the appearances of that of the sun, it only transfers to this star, and in a contrary direction, the real motion of the earth: the appearances are consequently the same, whether the earth be supposed to be at rest, or to revolve about the sun. In order to trace more particularly the identity of these appearances, let us conceive a radius drawn from the centre of the earth to that of the sun; this radius will be perpendicular to the plane which separates the enlightened from the darkened hemisphere of the earth. The sun is vertical to the point where it intersects the surface of the earth, and all the points of the terrestrial parallel, which this ray meets successively, in consequence of the diurnal motion have this star in their zenith at noon. But whether the sun revolves about the earth, or the earth about the sun and on its own axis, as it always preserves its parallel position, it is evident that this radius will trace the same surve on the surface of the earth; in each case it intersects the same terrestrial parallels. When the apparent longitude of the

sun is the same, this star will be equally elevated above the horizon, and the duration of the days will be equal. Thus, the seasons and the days are precisely the same, whether the sun be supposed to be at rest, or to revolve about (*d*) the earth; and the explanation of the seasons, which has been given in the preceding book, is equally applicable to the first hypothesis.

The planets all move in the same direction about the sun, but with different velocities; the durations of their revolutions increase in a greater ratio than their distances from this star; for instance, Jupiter employs nearly twelve years to perform its revolution, but the radius of the orbit is only five times greater than the radius of the earth's orbit; its real velocity is consequently less than that of the earth. This diminution of velocity in the planets according as they are more distant from the sun, obtains generally from Mercury, which is the nearest, to Uranus, which is the most remote from this star; and it follows, from the laws which we shall hereafter demonstrate, that the mean velocities of the planets are reciprocally as the square roots of (*e*) their mean distances from the sun.

Let us consider a planet of which the orbit is surrounded by that of the earth, and follow it from its superior to its inferior conjunction: its apparent or geocentric motion is the result of its real motion combined with that of the earth, estimated in a contrary direction. In the superior conjunction, the real motion of the planet is

contrary to that of the earth; therefore its geocentrick motion is then equal to the sum of these two motions, and it has the same direction as the geocentrick motion of the sun, which results from the motion of the earth transferred to this star in a contrary direction; consequently the apparent motion of the planet is direct. In inferior conjunction, the direction of the motion of the planet is the same as that of the earth, and as it is greater, the geocentrick motion preserves the same direction, which consequently is contrary to the apparent motion of the sun; therefore the planet is then retrograde. It is easy to conceive that in the passage from the direct to the retrograde motion, it must appear without motion, or stationary, and that this will happen between the greatest elongation and inferior *(f)* conjunction, when the geocentrick motion of the planet resulting from its real motion and that of the earth, applied in a contrary direction, is in the direction of the visual ray of the planet. These phenomena are entirely conformable to the motions that are observed to take place in the planets Mercury and Venus.

The motion of the planets, whose orbits comprehend that of the earth, has the same direction in their oppositions, as the motion of the earth, but it is less, and being combined with this last motion applied in a contrary direction, the direction of the motion which it assumes is opposed to its primitive direction, therefore in this position, the geocentrick motion of these planets is retrograde, it is direct in the conjunctions

like the motions of Venus and of Mercury, which are also direct in their superior conjunctions.

If the motion of the earth be transferred to the stars in a contrary direction, they must appear to describe in the interval of a year, a circumference equal and parallel to the terrestrial orbit, the diameter of which would subtend at the star, an angle equal to that under which the diameter of this orbit would appear (*g*) from their centre. This apparent motion is very similar to that which results from the combination of the motion of the earth with that of light, in consequence of which the stars appear to describe annually, a circumference parallel to the ecliptic, the diameter of which subtends an angle equal to 125″, but it differs from it in this, that in the first circumference the position of the stars is precisely the same as that of the sun, whereas in the second circumference they are less advanced than this star, by one hundred degrees. It is by means of this circumstance that we are able to distinguish between these two motions, and that we are assured that the first is at least extremely small, as the immense distance of the fixed stars renders the angle, which the diameter of the earth's orbit subtends when seen from this distance, almost insensible.

As the axis of the world is the prolongation of the axis of rotation of the earth, the motion of the poles of the celestial equator, indicated by the phenomena of precession and nutation (which have been explained in the XIIIth

Chapter of the first book), must be referred to this last axis. Therefore at the same time that the earth revolves on its axis and about the sun, its axis of rotation moves very slowly about the poles of the ecliptic, making very small oscillations, the period of which is the same as the motion of the nodes of the lunar orbit. Finally, this motion is not peculiar to the earth, for it has been observed in the IVth Chapter of the first book, that the axis of the moon moves in the same period, about the poles of the ecliptic.

CHAP. IV.

Of the laws of motion of the planets about the sun, and of the figure of their orbits.

NOTHING would be more easy than to calculate from the preceding data, the position of the planets at any given moment, if their motions about the sun were circular and uniform. But they are subject to very sensible inequalities, the laws of which constitute one of the most important objects of Astronomy, and the only clew which can conduct us to a knowledge of the general principle of the heavenly motions. In order to recognize these laws in the appearances which the planets present to us, we must divest their motions of the effects of the motion of the earth, and refer to the sun, their position as observed from different points of the earth's orbit. The dimensions of this orbit must be therefore first of all determined, and the law of the motion of the earth.

It has been shewn in the second Chapter of the first book, that the apparent orbit of the sun is an ellipse of which the earth occupies one of the foci, but as the sun is really immoveable, he should be placed in the focus, and the earth in the circumference of the ellipse. The motion of the

sun will be the same, and in order to obtain the position of the earth as seen from the centre of the sun, we should increase the position of that star, by two right angles.

It was also observed, that the sun appears to move in his orbit in such a manner that the radius vector, which connects its centre with that of the earth, traces about it areas proportional to the times in which they are described, but in reality these areas are traced about the sun. In general, every thing that has been stated in the chapter already cited, relative to the excentricity of the solar orbit and its variations, and respecting the position and motion of its perigee, may be also applied to the terrestrial orbit, with this sole exception, namely, that the earth's perigee is distant by two right angles from the perigee of the sun. The figure of the earth's orbit being thus known, let us examine how those of the other planets may be determined. For example, let us consider the planet Mars, which, from the great excentricity of its orbit, and its proximity to the earth, is peculiarly well adapted to make known the laws of the planetary motions.

The orbit of Mars and its motion about the sun would be known, if the angle which its radius vector makes with an invariable line passing through the centre of the sun be known at any instant, and also the length of this radius. In order to simplify the problem, we select those positions of Mars, in which one of these quan-

tities can be found separately; and this is very nearly the case in the oppositions, when the planet is observed to correspond to the same point of the ecliptic, to which it would be referred from the centre of the sun. From the difference between the angular motions of the earth and Mars, this planet corresponds to different points of the heavens in successive oppositions, therefore by comparing together a great number of observed oppositions, we are enabled to discover the law which exists between (*a*) the time and the angular motion of Mars about the sun, which is termed his *heliocentrick motion*. The different methods which are furnished by analysis, are considerably simplified in the present case, by considering that as the principal inequalities of Mars become the same at the termination of each sidereal revolution, their sum may be (*b*) expressed by a rapidly converging series of the sines of angles which are multiples of its mean motion, the coefficients of which series may be easily determined by means of some select observations.

The law of the radius vector of Mars may afterwards be obtained, by comparing observations of this planet made near its quadratures, in which case the angle which this radius subtends is the greatest. In the triangle formed by lines which join the centres of the earth, of the sun and of Mars, the angle at the earth is determined by direct observation, the law of the heliocentrick motion (*c*) of Mars, furnishes the angle at the sun, by means of which we may determine the radius vector of Mars in parts of

the radius of the earth, which is itself determined in parts of the mean distance of the earth from the sun. By comparing together a great number of radii vectores thus determined, the law of their variations corresponding to the angles which they make with an invariable right line, may be determined, by which means the figure of the orbit can be traced.

It was by a method very nearly similar, that Kepler discovered the lengthened form of the orbit of Mars; he conceived the fortunate idea of comparing its figure with that of an ellipse, the sun being in one of the foci; and the numerous observations of Tycho exactly represented in the hypothesis of an elliptic orbit, left no doubt as to the truth of this hypothesis.

The extremity of the greater axis of the orbit which is nearest to the sun, is called the *perihelion*, and the *aphelion* is the extremity which is farthest from the sun. The angular velocity of Mars about the sun is greatest at the perihelion; it diminishes according as the radius vector increases, and it is least at the aphelion. A comparison of this velocity with the powers of the radius vector, shews that it is reciprocally proportional to its square, from which it follows, that the product (d) of the daily helioecentrick motion of Mars, into the square of its radius, is constant. This product is double of the small vector, traced by its radius about the sun, therefore the area which it describes departing from an invariable line passing through the centre of the sun, increases as

the number of days which have lapsed since the epoch when the planet was upon this line ; consequently the areas described by the radius of Mars are proportional to the times. These laws of the motion of Mars, which have been discovered by Kepler, are the same as those of the apparent motion of sun, which have been developed in the second Chapter of the first book, they equally obtain in the case of the earth. It was natural to extend them to the other planets; Kepler therefore established as fundamental laws of the motions of these bodies, the two following, which all subsequent observations have fully confirmed.

The orbits of the planets are ellipses, of which the centre of the sun occupies one of the foci.

The areas described about this centre by the radii vectores of the planets, are proportional to the times of their description.

These laws are sufficient to determine the motion of the planets about the sun ; but besides it is necessary to know for each of them, seven quantities, which have been called the elements of *elliptic motion.* Five of these elements respect the motion in the ellipse, and are, 1st, the duration of the sidereal revolution ; 2d, the semiaxes major of the orbit, or the mean distance of the planet from the sun ; 3d, the excentricity, from which may be obtained the greatest equation of the centre ; 4th, the mean longitude of the planet at a given epoch ; 5th, the longitude of the perihelion at the same epoch. The two other elements are relative to the position of the orbit

itself; and are 1st, the longitude at a given epoch, of the nodes of the orbit, or of its points of intersection with a plane which is usually assumed to be that of the ecliptic. 2d, The inclination of the orbit to this plane. Therefore, for the seven planets which were known previous to the present century, there were forty-nine elements to be determined. The following table exhibits all those elements for the first instant of the present century, *i. e.* for the first of January, 1801, at midnight, according to the mean time of Paris.

The examination of this table shews that the durations of the revolutions of the planets increase with their mean distances from the sun. Kepler, for a long time, sought the relation which existed between the distances and periods; after a great number of trials, continued during sixteen years, he at length recognized that the squares *(e)* of the times of the planets' revolutions, are to each other as the cubes of the major-axes of their orbits.

Such are the fundamental laws of the planetary motion, which by exhibiting atronomy under a new aspect, have led to the discovery of universal gravitation.

The planetary ellipses are not invariable; their major axes appear to be always the same; but their excentricities, their inclinations to a fixed plane, the positions of their nodes and perihelions, are subject to variations, which hitherto appear to increase proportionally to the time. As *(f)* these variations do not become sensible until after the lapse of ages, they have been denominated *secular inequa-*

lities. There can be no doubt of their existence; but modern observations are not sufficiently removed from each other, nor are the ancient observations sufficiently exact to enable us to determine exactly their precise quantity.

There have been likewise observed *periodic* inequalities, which derange the elliptic motions of the planets. That of the earth is a little affected; for it has been before observed, that the apparent elliptic motion of the sun appears to be so. But these inequalities are principally apparent in the two larger planets, Jupiter and Saturn. From a comparison of ancient with modern observations, astronomers have inferred a diminution in the duration of Jupiter's revolution, and an increase in that of Saturn. A comparison of (*g*) modern observations with each other furnishes a contrary result; which seems to indicate in the motion of these planets, great inequalities of very long periods. In the preceding century, the duration of the revolutions of Saturn seemed to be different, according as the departure of the planet is supposed to take place from different points of its orbit; its returns to the vernal equinox, have been more rapid than to the autumnal. Finally, Jupiter and Saturn experience inequalities, which amount to several minutes, and which seem to depend on the situation of these planets, either among themselves, or with respect to their perihelions. Thus, every thing indicates that in the planetary system, independently of the principal cause which makes the planets to revolve in elliptic orbits

about the sun ; there exists several particular causes, which derange their motions, and at length change the elements of their ellipses.

TABLE OF THE ELLIPTIC MOTION OF THE PLANETS.

Durations of their sidereal revolutions.

		days.
Mercury	.	87, 96925804
Venus	.	224, 70078690
The Earth	.	365, 25638350
Mars .	.	686, 9796458
Jupiter	.	4332, 5848212
Saturn	.	10759, 2198174
Uranus	.	30686, 8208296

Semiaxes axes majores of their orbits, or their mean distances.

Mercury	.	.	0,3870981
Venus	.	.	0,7233316
The Earth	.	.	1,0000000
Mars .	.	.	1,5236923
Jupiter	.	.	5,202776
Saturn	.	.	9,5387861
Uranus	.	.	19,1823901

Ratio of the excentricities to the semiaxes majores at the commencement of 1801.

Mercury . .	0,20551494
Venus . .	0,00686074
The Earth . .	0,01685318
Mars . .	0,0933070
Jupiter . .	0,048162160
Saturn . .	0,05615051
Uranus . .	0,0466108

The mean longitude for the midnight, which separates the 31st of December, 1800, and the 1st of January, 1801, mean time at Paris.

Mercury . .	$182{,}^{\circ}15647$
Venus . .	11,93259
The Earth . .	111,28179
Mars . .	71,24071
Jupiter . .	124,68251
Saturn . .	150,35354
Uranus . .	197,55589

Mean longitude of the perihelion, at the same epoch.

Mercury . .	$82{,}^{\circ}6256$
Venus . .	143,0349
The Earth . .	110,5571
Mars . . .	369,3323
Jupiter . .	12,3810
Saturn . .	99,0647
Uranus . .	186,1500

The sidereal and secular motion of the perihelion, (the sign — indicates a retrogade motion.)

Mercury . .	1801,22
Venus . . .	826,76
The Earth . .	3646,61
Mars	4882,70
Jupiter . .	2054,44
Saturn . .	5978,67
Uranus . .	740,98

The inclination of the orbit to the ecliptic at the commencement of 1801.

Mercury . .	$7^{\circ},78058$
Venus . . .	3,76807
The Earth . .	0,00000
Mars . . .	2,05746
Jupiter . . .	1,46029
Saturn . . .	2,77027
Uranus . .	0,86069

The secular variation of the inclination to the true ecliptic, (the sign — indicates a diminution.)

Mercury . . .	$56'',12$
Venus . . .	14,05
The Earth . .	0,00
Mars . . .	0,81
Jupiter . . .	69,77

	″
Saturn . . .	47,88
Uranus . . .	9,73

The longitude of the ascending node at the commencement of 1801.

	°
Mercury . .	51,0651
Venus . . .	83,2262
The Earth . .	0,0000
Mars . . .	53,3344
Jupiter . . .	109,3762
Saturn . . .	124,3819
Uranus . .	81,1035

The sidereal and secular motion of the node on the true ecliptic (the signs — indicates a retrograde motion,)

	″
Mercury . .	2414,39
Venus . . .	5775,92
The Earth . .	0,00
Mars . . .	7187,50
Jupiter . .	4880,97
Saturn . . .	5995,35
Uranus . .	11107,43

The elements of the orbits of the four planets recently discovered cannot be yet obtained with precision, as the time during which they have been observed has been very short; besides the consi-

derable perturbations which they experience, have not as yet been determined. Underneath are presented the elliptic elements which best satisfy the observations hitherto made, but they ought only to be considered as a first sketch of the theory of the planets.

Durations of their sidereal revolutions.

	days.
Ceres . .	1681,3931
Pallas . .	1686,5388
Juno . . .	1592,6608
Vesta . .	1325,7431

Semi axes-majores of their orbits.

Ceres . . .	2,767245
Pallas . .	2,772886
Juno . . .	2,669009
Vesta . .	2,36787

Ratio of the excentricity to the semiaxis major.

Ceres . . .	0,078439
Pallas . .	0,241648
Juno . . .	0,257848
Vesta . . .	0,089130

Mean longitude at the midnight commencing 1820.

Ceres . . .	136,8461
Pallas . . .	120,3422
Juno . . .	222,3989
Vesta . . .	309,2917

Longitude of the perihelion at the same epoch.

Ceres . . .	163, 4727
Pallas . . .	134, 5754
Juno . . .	59, 5142
Vesta . . .	277, 2853

Inclination of the orbit to the ecliptic.

Ceres . . .	$11^{\circ},8044$
Pallas . . .	38,4344
Juno . . .	11,5215
Vesta . . .	7,9287

Longitude of the ascending node at the commencement of 1810

Ceres . . .	87°,6557
Pallas . . .	191,8416
Juno . . .	190,1421
Vesta . . .	114,6908

CHAP. V.

Of the figure of the orbits of the comets, and of the laws of their motion about the sun.

THE sun being at the focus of the planetary orbits, it is natural to suppose that he is also in the focus of the orbits of the comets. But as these stars disappear after having been visible some months at most, their orbits, instead of being nearly circular, like those of the planets, are very excentric, and the sun is very near to that part in which they are visible. The ellipse, by means of the infinite varieties which it admits of from the circle to the parabola, may represent these different orbits. Analogy leads us then to suppose that the comets move in ellipses, of which the sun occupies one of the foci, and to consider them as moving according to the same laws as the planets, so that the areas traced by their radii vectores are equal in equal times.

It is almost impossible to know the duration of the revolution of a comet, and consequently the greater axis of its orbit, by an observation of only one of its appearances; hence the area which its radius vector describes in a given time, cannot be determined rigorously. But it should be considered that the small portion of the ellipse,

described by the comet during its appearance, may be supposed to coincide with a parabola, and that thus its motion may be calculated in this interval, as if it was parabolical.

According to the laws of Kepler, the sectors, traced in equal times by the radii vectores of two planets, are to each other as the areas of (a) their ellipses, divided by the times of their revolutions; and the squares of these times are to each other as the cubes of their greater semiaxes. It is easy to infer from this, that if a planet be supposed to move in a circular orbit, of which the radius is equal to the perihelion distance of the comet, the sector, described by the radius vector of the comet, will be to the corresponding sector described by the radius vector of the planet, in the ratio of the square root of the aphelion (b) distance of the comet to the square root of the semiaxis major of its orbit, which ratio, when the ellipse changes into a parabola, becomes that of the square root of two to unity; by this means, the ratio of the sector of the comet to that of the fictitious planet may be obtained; and it is easy by what precedes to obtain the ratio of this sector to that which the radius vector of the earth traces in the same time. We can therefore determine for any instant whatever, the area traced by the radius vector of the comet, commencing with the moment of its passage through the perihelion, and fix its position in the parabola, which it is supposed to describe.

Nothing more is necessary but to determine, by means of observations, the elements of the parabolick motion, that is to say, the perihelion distance of the comet, in parts of the mean distance of the sun from the earth, the position of the perihelion, the instant of the passage through the perihelion, the inclination of the orbit to the ecliptic, and the position of its nodes. The investigation of these five elements presents greater difficulties than that of the elements of the planets, which being always visible, may be compared in positions the most favourable to the determination of these elements, while on the other hand, the comets are only visible for a short time, and almost always in circumstancs, in which their apparent motion is extremely complicated by the real motion of the earth, which we transfer to them in a contrary direction. Notwithstanding all these obstacles, we have succeeded by different methods, in determining the orbits of the comets. Three complete observations are more than sufficient for this purpose; all the others serve only to confirm the accuracy of these elements, and the truth of the theory which we have explained. More than one hundred comets, of which the numerous observations, are exactly represented by this theory, remove all doubt as to its accuracy. Thus, the comets, which for a long time were regarded as meteors, are stars similar to the planets; their motions and their returns are regulated by laws similar to those which influence the planetary motions.

Let us take notice here, how the true system of nature, according as it developes itself, receives more confirmation. The simplification of the celestial phenomena, on the hypothesis that the earth moves, compared with their great complexity, on the hypothesis of its immobility, renders the first of these hypotheses extremely probable. The laws of elliptic motion, common then to the planets and to the comets, increase this probability considerably, which becomes still greater from the consideration that the motions of the comets are subject to the same laws.

These stars do not all move in the same direction, like the planets. Some have an actual direct motion, the direction of the motion of others is retrograde; the inclinations of their orbits are not confined within a narrow zone like those of the planetary orbits; they exhibit all varieties of inclination, from the orbit situated on the plane of the ecliptic, to an orbit perpendicular to this plane.

A comet is recognized when it reappears, by the identity of the elements of its orbit with those of the orbit of a comet already observed. If the perihelion distance, the position of this perihelion and of its nodes, and the inclination of its orbit be very nearly the same, it is then extremely probable that the comet which appears is that which had been observed before, and which after having receded to such a distance that it was invisible, returns into that part of its orbit which is nearest to the sun. As the durations of the revo-

lutions of comets are very long, and as it is not quite two centuries since these stars have been carefully observed, the period of the revolution of only one comet is known with certainty, namely, that of 1759, which had been before observed in 1682, 1607, and 1531. This comet returns to its perihelion in about seventy-six years. Therefore if the mean distance of the sun from the earth be assumed equal to unity, the greater axis of its orbit is very nearly 35,9 ; and as its perihelion distance is only 0,58, it recedes from the sun, at least thirty-five times farther than the earth, describing an extremely excentric ellipse. Its return to the perihelion was longer by thirteen months from 1531 to 1607, than from 1607 to 1682 ; and it was eighteen months shorter, from 1607 to 1682, than from 1682 to 1759. It appears therefore that causes similar to those which derange the elliptic motion of the planets, disturb also that of the comets in a much more sensible manner.

The return of some other comets has been suspected ; the most probable of these returns was that of the comet of 1532, which was supposed to be the same with that of 1661, the time of the revolution of which has been fixed at 129 years, but this comet not having appeared in 1790, as was expected, there is great reason to believe that these two comets were not the same, and we shall not be surprized at this, if we consider the inaccuracy of the observations of Appian and Frucastor, from which the elements were determined in 1532.

These observations are so rude, that according to Mechain, who has carefully examined them, they leave an uncertainty of 41°, on the position of the node, of 10°, on the inclination, of 22°, on the position of the perihelion, and of 0,255 on the perihelion distance.

The clements of the orbit of the comet observed in 1818, correspond so exactly with those of the orbit of the comet observed in 1805, that it has been inferred that these comets are the same, which would assign the short period of thirteen years for the time of revolution, provided that there was no intermediate return of the comet to its perihelion; but M. Enk has ascertained by a careful discussion of the numerous observations of this star in 1818 and 1819, that its revolution is still less by 1203^d very nearly; he concluded that it should reappear in 1822: and in order to facilitate to observers the means of finding it again, he computed the position which it aught to have on each day of its approaching appearance. From the southern declinations of the comet during the time of this appearance, it is almost impossible to observe it in Europe. Fortunately it has been observed at New Holland by M. Rumker, an expert Astronomer, who was brought there by General Brisbane, who is himself an able Astronomer, and has interested himself very much in the advancement of this science. M. Rumker observed it for each successive day, from the 2d to the 23d of June 1822, and its observed positions accord so well with those which M. Enk had previously

calculated, that there cannot remain any doubt on this return of the comet, predicted by M. Enk.

The nebulosity with which the comets are almost always surrounded, seems to be formed by the vapours which the solar heat excites from their surface. In fact, the great heat which they experience near to their perihelion, may be supposed to rarify the particles which have been congealed by the excessive cold of the aphelion. This heat is most intense for those comets, whose perihelion distance is very small. The perihelion distance of the comet of 1680, was one hundred and sixty-six times less than the distance of the sun from the earth, and consequently it ought to experience a heat twenty-seven thousand five hundred times greater than that which is communicated to the earth, if, as (*d*) every thing induces us to think, the heat is proportional to the intensity of its light. This excessive heat, which is much greater than any which we could produce, would volatilise, according to all appearances, the greater number of terrestrial substances.

Whatever be the nature of heat, we know that it dilates all bodies. It changes solids into fluids, and fluids into vapours. These changes of form are indicated by certain phenomena which we will trace from ice. Let us consider a volume of snow or of pounded ice in an open vessel submitted to the action of a great heat. If the temperature of this ice be below that of melting ice, it will increase up to zero of temperature. After

having attained this (*e*) point, the ice will melt by new additions of heat; but if care be taken to agitate it, until all the ice is converted, will always remain at the same temperature, and the heat communicated by the vessel will not be sensible to the thermometer immersed in it, as it will be entirely occupied in converting the ice into water. After all the ice is melted, the additional heat will continually raise the temperature of the water and of the thermometer till the moment of ebullition. The thermometer will then become stationary a second time; and the heat communicated by the vessel will be entirely employed in reducing the water into steam, the temperature of which will be the same as that of boiling water. It appears from this detail, that the water produced by the melting of ice and the vapours into which boiling water is converted, absorb at the moment of their formation a considerable quantity of caloric, which reappears in the reconversion of aqueous vapours to the state of water, and of water to the state of ice; for these vapours, when condensed on a cold body, communicate much more heat to it than it would receive from an equal weight of boiling water; besides we know that water can preserve its fluidity, though its temperature may be several degrees below zero; and that in this state, if it is slightly agitated, it is converted into ice, and the thermometer, when plunged in it, ascends to zero, in consequence of the heat given out during

this change. All bodies which we can make pass from a solid to a fluid (*f*) state, present similar phenomena; but the temperatures at which their fusion and ebullition commences, are very different for each of them.

The phenomenon which has been just detailed, although very universal, is only a particular case of the following general law, "*in all the changes of condition which a body undergoes from the action of caloric, a part of this caloric is employed in producing them, and becomes latent,* that is to say, insensible *to the thermometer; but it reappears when the system returns to its primitive state.*" Thus when a gas contained in a flexible envelope is dilated by an increase of temperature, the thermometer is not affected by the part of the caloric which produces this effect, but this latent part becomes sensible when the gaz is reduced by compression to its original density.

There are bodies which cannot be reduced to a state of fluidity, by the greatest heat which we can produce. There are others which the greatest cold experienced on earth is unable to reduce to a solid state: such are the fluids which compose our atmosphere, and which, notwithstanding the pressure and cold to which they have been subjected, have still maintained themselves in the state of vapours. But their analogy with aeriform fluids, to which we can reduce a great number of substances bythe application of heat, and their condensation by compression and cold, leaves no doubt but that the atmospheric fluids are extremely

volatile bodies, which an intense cold would reduce to a solid state. To make them assume this state, it would be sufficient to remove the earth farther from the sun, as it would be sufficient in order that water and several other bodies should enter into our atmosphere, to bring the earth nearer to the sun. These great vicissitudes take place in the comets, and principally on those which approach very near to the sun in their perihelion. The nebulosities which surround them, being the effect of the vaporisation of fluids at their surface, the cold which follows ought to moderate the excessive heat which is produced by their proximity to the sun; and the condensation of the same vaporised fluids when they recede from it, repairs in part the diminution of temperature, which this remotion ought to produce, so that the double effect of the vaporisation and condensation of fluids, makes the difference between the extreme heat and cold, which the comets experience at each revolution, much less than it would otherwise be.

When the comets are observed with very powerful telescopes, and under circumstances in which we ought only to perceive a part of the illuminated hemisphere, we are not able to discover any phases. One only, comet namely, that of 1682, presented them to Hevelius and La Hire.

We shall see in the sequel, that the masses of the comets are extremely small, the diameters of their disks must therefore be nearly insensible, and what is termed their *nucleus* is most

probably made up in a great part, of the densest strata of the nebulosity which surrounds them. Thus Herschel has discovered by means of very powerful telescopes in the nucleus of the comet of 1811, a brilliant point which he judged with reason to be the disk of the comet. These strata are extremely rare, in as much as the stars have been sometimes observed through them. It appears that the tails which accompany the comets, are formed by the most volatile particles, which are excited at their surface by the heat of the sun, and which are dispersed indefinitely by the impulsion of its rays. This may be inferred from the direction of these trains of vapour, which are always beyond the comet relatively to the sun, and which continually increasing according as these stars approach to this luminary, do not attain their maximum till after these bodies have passed through the perihelion. From the extreme tenuity of the molecules, the ratio of the surfaces to the masses is increased, so that it may render sensible the impulsion of the solar rays, (*g*) which ought then to make each particle to describe an hyperbolic orbit, the sun being in the focus of the corresponding conjugate hyperbola. A succession of molecules moving on these curves from the head of the comet, form a luminous train directed from the sun, and forming a small angle with that part of the comet's orbit which it has passed over; this is in fact what observation indicates. From the quickness with which these tails increase, we may form some estimate of the rapidity of ascension of

their molecules. It is possible to conceive that differences of volatility, of magnitude, and of density, in the molecules, may produce considerable differences in the curves which they describe, which must cause great varieties in the form, the length, and the magnitude of the tails of the comets. If these effects be combined with those which may arise from a rotatory motion in these stars, and from the illusions of the annual parallax, we may be able to account for the singular appearances which their nebulosities and tails exhibit to us.

Although the dimensions of the tails of the comets may be several millions of myriametres, still they do not sensibly dim the light of the stars, which are seen through them ; they are therefore extremely rare, and it is probable that their masses are less than those of the smallest mountains of the earth. Consequently in the event of their meeting with this planet, they cannot produce any sensible effect. It is extremely probable that they have several times enveloped it without its being observed. The state of the atmosphere has a considerable influence on their apparent length and magnitude; between the tropics they appear much greater than in our climates. Pingre states, that he observed a star which appeared in the tail of the comet of 1769, and which receded from it in a few moments. But this appearance is only an illusion, which is produced by the clouds floating in our atmosphere, which are sufficiently dense to intercept the feeble light of this tail, at

the same time that they are sufficiently rare to enable us to perceive the more vivid light of the star. It cannot *(h)* be supposed that the molecules of the vapours of which the tails are composed, make such rapid oscillations, of which the extent surpasses a million of myriametres.

If the evaporable substances of a comet diminish at each of its returns to the perihelion, they ought after several revolutions to be entirely dissipated in space, and the comet ought only to exhibit afterwards the appearance of a solid neucleus; those comets whose revolution is short, will arrive at this state sooner than others. The comet of 1682, the time of whose revolution is only seventy-six years, is the only one which has as yet exhibited appearances which correspond to this state of fixity. If the neucleus be too small to be perceived, or if the evaporable substances which remain at its surface, are in too small a quantity to constitute by their evaporation, a sensible head to the comet; the star will be for ever invisible. Perhaps this is one of the reasons, which renders the reappearances of the comets so rare; perhaps it is on this account that the comet of 1770 has totally disappeared, though during the time of its appearance it described an ellipse in a period of five years and a half; so that if it has continued to describe this curve, it must since that epoch have returned at least five times to its *(i)* peri-

helion. Perhaps finally this is the cause, why several comets whose routs we can trace in the heavens by means of the elements of their orbits, have disappeared sooner than might be expected.

CHAP. VI.

Of the laws of the motion of the satellites about their respective primary planets.

We have explained in the sixth chapter of the first book the laws of the motion of the satellite of the earth, it now remains to consider those of the motions of the satellites of Jupiter, of Saturn, and of Uranus.

If the semidiameter of the equator of Jupiter, which is supposed to be $56'',702$ at the mean distance of Jupiter from the sun, be assumed equal to unity, the mean distances of the satellites from its centre and the durations of their sidereal revolutions will be (*a*) as follows:

	Mean distances.	Durations.
		days.
I. Satellite	6,04853	1,769137788148
II. Sat. . . .	9,62347	3,551181017849
III. Sat. . . .	15,35024	7,154552783970
IV. Sat. . . .	26,99835	16,688769707084

It is easy to infer the durations of the synodic revolutions of the satellites, or the intervals between the return of their mean conjunctions with Jupiter, from the durations of their sidereal revolutions, and from that of the revolution of Ju-

piter. From a comparison of their mean distances with the durations of their sidereal revolutions, it appears that the same beautiful proportion which has been observed to obtain between the durations of the revolutions of the planets and their mean distances from the sun, obtains also in the case of the satellites, namely, that the squares of the times of the sidereal revolutions of the satellites are as the cubes of their mean distances from the centre of Jupiter. The frequent recurrence of the eclipses of Jupiter's satellites, has furnished astronomers with the means of tracing their motions with a precicision, which could not be obtained by observing their angular distance from Jupiter. They have enabled us to recognize the following results :

The ellipticity of the orbit of the first satellite is insensible; its plane coincides very nearly with the plane of Jupiter's equator, the inclination of which to the plane of the orbit is about 4°,4352.

The ellipticity of the orbit of the second satellite is also insensible, its inclination to Jupiter's orbit is variable, as is also the position of its nodes. All these variations may be very nearly represented, by supposing the orbit of the satellite to be inclined in an angle of 5152″ to the equator of Jupiter, and by making its nodes to move on this plane with a retrograde motion, of which the period is thirty Julian years.

A slight ellipticity has been observed in the orbit of the third satellite, the extremity of its greater axis which is nearest to Jupiter, and which has been termed its *perigove*, has a direct motion,

but of a variable quantity. The excentricity of the orbit is also subject to very sensible variations near the close of the last century, the equation of the centre had attained its *maximum*, and amounted to 2458″ very nearly: it afterwards diminished, and near to 1777 it was at its *minimum*, when it amounted to 949″. The inclination of the orbit of this satellite to that of Jupiter, and the position of its nodes are variable; all these variations may be very nearly represented, by supposing the orbit to be inclined at an angle of 2284″, to the equator of Jupiter, and by assigning to its nodes a retrograde motion on the plane of this equator, in a period of 142 years. Notwithstanding this, astronomers who have determined by the eclipses of this satellite (*b*) the inclination of the equator of Jupiter on the plane of its orbit have found that it is invariably nine or ten minutes less than what is assigned by the eclipses of the first and of the second satellite. The orbit of the fourth satellite has a very visible ellipticity; its perigove moves in consequentia with an annual motion amounting to 7939″. The inclination of this orbit to that of Jupiter is about 2°,7. It is in consequence of this inclination, that the fourth satellite passes frequently behind the planet relatively to the sun without being eclipsed. From the discovery of the satellites until the year 1760, the inclination appeared to be constant, and the annual motion of the nodes on the orbit of Jupiter, has been direct and about 788″. But, since 1760, the inclination has increased, and the mo-

tion of the nodes has diminished in a very sensible manner. We shall resume the consideration of these inequalities, after their cause shall have been explained.

Independently of these variations, the satellites are subject to inequalities, which derange their elliptic motions, and render their theory extremely complicated. They are principally sensible in the three first satellites, of which the motions exhibit very remarkable relations.

It appears from a comparison of the times of their revolutions, that the period of the first satillite is only about half the duration of the period of the second, which itself is only half of that of the period of the third satellite. Thus, the mean motions of these three satellites follow very nearly a geometric progression, of which the ratio is one half. If this proportion obtained accurately, the mean motion of the first satellite, plus twice the mean motion of the third, would be precisely equal to three times the mean motion of the second. But this equality is much more accurate than the progression itself; so that we are induced to consider it as rigorously true, and to reject the very small quantities by which it deviates from it, as arising from the errors of observation; at least we can affirm that it will subsist for a long series of ages.

A result which is equally remarkable, and which is given by observation with equal precision, is, that from the discovery of the satellites, the mean longitude of the first minus three times that of

the second, plus twice that of the third, does not differ from two right angles, by any perceptible quantity.

These two results also obtain, between the mean motions, and the mean synodic longitudes; for as the synodic motion of a satellite is the excess of its sidereal motion above that of the planet, if in the preceding results, the synodic motions be substituted in place of the sidereal motions, the mean motion of Jupiter disappears, and these results remain the same. It follows from this, that for a great number of years at least, the three first satellites of Jupiter cannot be eclipsed together, but in the simultaneous eclipses of the first and third, the first will be always in conjunction with Jupiter; it will be always in opposition, in the simultaneous eclipses of the sun, produced at Jupiter by the two other satellites.

The periods and the laws of the principle inequalities of these satellities are the same. The inequality of the first advances or retards its eclipses, by 223″,5 of time at its *maximum*. A comparison of its quantity, in the respective positions of the two first satellites, shews that it disappears when these satellites seen from the centre of Jupiter, are at the same time, in opposition to the sun; that it afterwards increases, and becomes the greatest possible, when the first satellite at the moment of its opposition is 50° more advanced than the second; that it vanishes again when it is more advanced by 100 than the second, and that beyond this, it is affected with a contrary sign

and retards the eclipses; that it increases until the distance of the planets from each other is 150°, when it is at its negative *maximum*; that then it diminishes, and disappears when this distance is 200°; finally, in the second half of the circumference, it runs through the same series of changes as in the first. From this it has been inferred, that there exists in the motion of the first satellite about Jupiter, an inequality, which at its maximum is 5050″,6 of a degree, and proportional to the sine of double of the excess of the mean longitude of the first satellite above that of the second, which excess is equal to the difference of the mean synodic longitudes of the two satellites. The period of this inequality is only four days; but how is it transformed in the eclipses of the first satellite into a period of $437^d,6592$? this is what we proceed to explain.

Suppose that the first and second satellite depart together from their mean oppositions with the sun. After the description of each circumference, the first satellite will be, in virtue of its mean synodic motion, in its mean opposition with the sun. If we suppose an imaginary star, of which the angular motion is equal to the excess of the mean synodic motion of the first satellite, above twice that of the second; then twice the difference of the mean synodic motions of the two satellites will be, in the eclipses of the first, equal to a multiple of the circumference plus the motion of the imaginary star; consequently the sine of this last motion will be proportional to the in-

equality of the first satellite in the eclipses, and may represent it. Its period is equal to the duration of the revolution of the imaginary star, which duration is, from the mean synodic motions of the two satellites about $437^d,6592$; it is thus determined with greater accuracy than by direct observation.

The law of the inequality of the second satellite, is precisely the same as that of the first, with this difference, that it is always of a contrary sign; at its *maximum* it advances or retards the eclipses by about $1059'',2$ of a degree; from a comparison of the respective positions of the two satellites, it appears that it vanishes when they are at the same time in opposition to the sun; that it then retards the eclipses of the second more and more, until those two satellites are at the moment of the occurrence of the phenomena, elongated from each other one hundred degrees; that this retardation diminishes and becomes nothing a second time, when the mutual distance of the two satellites is two hundred degrees; finally, that beyond this time, the eclipses advance as they had previously retarded. From these observations it has been inferred, that there exists in the motion of the second satellite, an inequality of $11920'',7$ at its *maximum*, and that it is proportional to, and affected with a contrary sign, to the sine of the mean longitude of the first satellite over that of the second, which excess is equal to the difference of the mean synodic motions of the two satellite.

If the two depart together from their mean opposition to the sun, the second will be in its mean opposition after the completion of each circumference, which it will have described in consequence of its mean synodic motion. If, as in the case of the first satellite, we conceive a star of which the angular motion is equal to the excess of the mean synodic motion of the first satellite, over twice that of the second, then the difference of the mean synodic motions of the two satellites will be, in the eclipses of the second equal to a multiple of the circumference, plus the motion of the fictitious star; consequently the inequality of the second satellite will, in its eclipses, be proportional to the sine of the motion of this imaginary star. Hence we see the reason why the period and the law of this inequality, are the same, as those of the inequality of the first satellite.

The influence of the first satellite, on the inequality of the second is very probable. But if the third produces in the motion of the second, an inequality similar to that which the second seems to produce in the motion of the first, that is to say, proportional to the sine of double of the difference of the mean longitudes of the second and third satellite; this new inequality will be confounded with that which arises from the first satellite, for in consequence of the relation which exists between the mean longitudes of the three first satellites, and what has been already explained, the difference of the mean longitudes of the two first satellites is equal to the same circumfer-

ence plus, twice the difference between the mean longitudes of the second and third satellites, so that the sine of the first difference is the same as the sine of twice the second difference, only affected with a contrary sign. The inequality produced by the third satellite, in the motion of the second, would thus have the same sign, and would follow the same law as the inequality observed in this motion; it is therefore extremely probable that this inequality is the result of two inequalities depending on the first and third satellite. If in the progress of time, the preceding relation between the longitudes should cease to exist; these two inequalities which are now blended together would be separated, and we might by observation determine their respective values. But we have seen that this relation must subsist for a very long time, and in the fourth book it will appear, that this relation is rigorously true. Finally, the inequality relative to the third satellite in its eclipses, compared with the respective positions of the second and third, presents the same relations as the inequality of the second, compared with the respective positions of the two first satellites. Consequently there exists in the motion of the third satellite, an inequality proportional to the sine of the excess of the mean longitude of the second satellite above that of the third, which inequality at its *maximum* is 808″, of a degree. If we conceive a star of which the angular motion is equal to the excess of the mean syno-

dic motion of the second satellite above twice the mean synodic motion of the third, the inequality of the third satellite in the eclipses will be proportional to the sine of the motion of the imaginary star; but in consequence of the relation which subsists between the mean longitudes of the three first satellites, the sine of this motion is with the exception of the sign, the same as that of the motion of the first imaginary star which has been considered. Thus the inequality of the third satellite in its eclipses has the same period, and follows the same laws, as'the inequalities of the two first satellites.

Such are the periods of the principal inequalities of the three first satellites of Jupiter, which Bradley seems to have suspected, but which Vargenten has since detailed with the greatest accuracy. Their correspondence and that of the mean motions and mean longitudes of these satellites, appear to constitute a separate system of these three bodies, actuated according to all appearance by common forces, from which arise those relations, which they have in common.

If the apparent semidiameter of the equator of this planet, at its mean distance from the sun, which is about 25′′, be assumed as unity, the mean distances of the satellites from its centre, and the durations of their sidereal revolutions are:

	Mean distances.	Durations.
		d
I.	3,351	0,94271.
II.	4,300	1,37024.
III.	5,284	1,88780.
IV.	6,819	2,73948.
V.	9,524	4,51749.
VI.	22,081	15,94530.
VII.	64,359	79,32960.

By comparing the durations of the revolutions of the satellites, with their mean distances from the centre of Saturn, we recognize the beautiful relation discovered by Kepler relatively to the planets, and which we have already observed to exist in the system of the satellites of Jupiter, *i. e.* that the squares of the times of the revolutions of the satellites of Saturn, are as the cubes of their mean distances from the centre of this planet.

The great distance of the satellites of Saturn, combined with the difficulty of observing their position, has not enabled us to recognize the ellipticity of their orbits, and still less the inequalities of their motions. However, the ellipticity of the orbit of the sixth satellite is perceptible.

If we assume as unity, the semidiameter of Uranus, which is 6″, when seen from the mean distance of the planet from the sun; the mean distances of the satellites from its centre, and the durations of their revolutions are, according to the observations of Herchell :

	Mean distances.	Durations.
I.	13,120	5,8926^{d}.
II.	17,022	8,7068.
III.	19,845	10,9611.
IV.	22,752	13,4559.
V.	45,507	38,0750.
VI.	91,008	107,6944.

These durations, with the exception of the second and fourth, have been inferred from the greatest observed elongations, and from the law according to which the squares of the periods are proportional to the cubes of their mean distances from the planet, which law is confirmed by observations made on the second and fourth satellite, the only ones which are sufficiently well known; so that it may be considered as a general law of the motion of a system of bodies which revolve about a common focus.

It may now be asked what are the principal forces which retain the planets, the satellites and the comets in their respective orbits? what particular forces derange their elliptic motions; what cause makes the equinoxes to regrade, and produces the rotation of the earth and moon about their axes; finally, by the action of what forces, are the waters of the sea raised twice each day; the supposition of one sole principle on which all these laws depend, is worthy of the majestic simplicity which pervades all nature. The generality of the laws which the

celestial motions present, seems to indicate its existence; even already we may suspect that such a principle is in existence, from the connection between these phenomena and the respective positions of the bodies of the solar system. But in order that we may be able to place it in the clearest light, the laws of the motion of matter must be known.

BOOK THE THIRD.

OF THE LAWS OF MOTION.

At nunc per maria ac terras sublimaque cæli
Multa modis multis varia ratione moveri
Cernimus ante oculos.

LUCRET: lib. 1.

SURROUNDED as we are by an infinite variety of phenomena, which continually succeed each other in the heavens and on the earth, philosophers have succeeded in recognizing the small number of general laws to which matter is subject in its motions. To them, all nature is obedient; and every thing is as necessarily derived from them, as the return of the seasons; so that the curve which is described by the lightest atom that seems to be driven at random by the winds, is regulated by laws as certain as those which confine the planets to their orbits. The importance of these laws, on which we continually depend, ought to have excited the curiosity of mankind in all ages; yet by the effect of an indifference but too common to the human mind, they were utterly unknown, until the commencement of the

17th century, at which epoch Gallileo laid the first foundations of the science of motion by his beautiful discoveries on the descent of bodies. Geometricians, following up the steps of this great man, have finally reduced the whole science of mechanics to general formula, which leaves nothing to be desired but to bring the analysis to perfection.

CHAP. I.

Of forces, of their composition, and of the equilibrium of a material point.

A BODY appears to us to move, when it changes its situation relatively to a system of bodies which we suppose to be at rest. Thus in a vessel which moves in an uniform manner, bodies seem to us move, when they correspond successively to its different parts. This motion is only relative; for the vessel moves on the surface of the sea, which revolves round the axis of the earth, the centre of which moves about the sun, which is itself carried along in the regions of space, together with the earth and the planets. In order to conceive a term to those motions, and to arrive at length at some fixed points, from which we may reckon the absolute motion of bodies, we conceive a space without bounds, immoveable, and penetrable to matter; and it is to different parts of this space, whether real or imaginary (*a*) that we in imagination refer the position of bodies; and we conceive them to be in motion when they correspond successively to different points of this space.

The nature of that singular modification, in consequence of which a body is transported from

one place to another, is, and always will be, unknown. It has been designated by the name of *Force;* but we can only determine its effects and the law of its action.

The effect of a force acting on a material point, is, if no obstacle intervenes, to put it in motion. The direction of the force is the right line, which it tends to make the point described. It is evident that if two forces act in the same direction, their effect to move the point is the sum of the forces, and that when they act in opposite directions, the point is moved by a (*b*) force represented by their difference, so that if the forces were equal, the point would remain at rest.

If the directions of the two forces make an angle with each other, a force results, the direction of which is intermediate between the directions of the composing forces, and it can be demonstrated by geometry alone, that if from the point of concourse of these forces, and in their respective directions, right lines be assumed (*c*) which represent them, and if then the parallellogram of which these lines are adjacent sides, be completed, its diagonal will represent their resultant, both in quantity and in direction. We may substitute in place of the two composing forces, their resultant, and conversely we can in place of any force whatever, substitute two others, of which it is the resultant, consequently any force may be resolved into two others parallel to two axes perpendicular to each other, and situated in a plane which passes through its direction. To

effect this, it is sufficient to draw through the first extremity of the line which represents this force, two lines respectively parallel to these axes, and to form on these lines a rectangle, the diagonal of which represents the force to be decomposed. The two sides of the rectangle will represent the forces, into which the given force may be decomposed parallel to these axes.

If the force be inclined to a plane given in position, then by assuming to represent it, a line in its direction, the extremity of which is in the point where it meets the plane ; the perpendicular demitted from the extremity of this line on the plane, will be the primitive force resolved perpendicularly to this plane. The right line drawn in this plane, connecting the line representing the given force and the perpendicular, will be the primitive force, decomposed parallel to the plane. This second partial force may itself be resolved into two others parallel to two axes situated in this plane, and at right angles to each other. Consequently every force may be resolved into three others, respectively parallel to three axes perpendicular to each other.

Hence arises a simple method of obtaining the resultant force of any number of forces, which act on a material point ; for by resolving each of them into three others parallel to three axes given in position, and at right angles to each other, it is evident that all the forces parallel to the same axis are reducible to a sin-

gle force, equal to the sum of those which act in one direction, minus the sum of those which act in a contrary direction. Consequently the point will be acted on by three forces, perpendicular to each other; if then three right lines in their respective directions be assumed to represent them, reckoning from their point of concourse, and a rectangular parallelopiped be formed on these three lines, its diagonal will represent the quantity and direction of (*d*) the force resulting from all those which act on the point.

Whatever may be the number, the magnitude, and the directions these forces, if the position of the point be varied in any manner by an indefinitely small quantity, the product of the resultant into the quantity by which the point advances in its direction, is equal to the sum of the products of each force into the corresponding quantity. The (*e*) quantity by which the point advances in the direction of any force, is the projection of the line connecting the two positions of the point, on the direction of the force; if the point advances in the opposite way from this direction, this quantity should be taken negatively.

In a state of equilibrium the resultant of all the forces vanishes, provided the point be free. If it is not, the resultant should be perpendicular to the surface, or to the curve on which the point is constrained to exist; and then, when the position of the point is changed by an indefinitely

small quantity, the product of the resultant into the quantity by which the point advances in its direction, vanishes ; this product is therefore always equal to (*f*) nothing, whether the point be supposed to be altogether free, or whether it be constrained to exist on a curve or surface. Consequently in all cases, in which the equilibrium obtains, the sum of the products of each force, into the quantity by which it advances in its direction, when an indefinitely small change is made in its position, vanishes ; and if this condition is satisfied, the equilibrium subsists.

CHAP. II.

Of the motion of a material point.

A Point in repose cannot excite any motion in itself, because there is nothing in its nature to determine it to move in one direction in preference to another. When sollicited by any force, and then abandoned to itself, it will move constantly and uniformly in the direction of that force, if it meets with no resistance; that is to say, at every instant its force and the direction of its motion are the same. This tendency of matter, to persevere in its state of rest or of motion, is what is termed its *inertia*; it is the first law of the motion of bodies.

The direction of the motion in a right line, follows necessarily from this, that there is no reason why the body should deviate to the right, rather than to the left of its primitive direction; but the uniformity of its motion is not equally evident. The nature of the moving force being unknown, it is impossible to know *a priori*, whether this force should continue without intermission or not. Indeed, as a body is incapable of exciting any motion in itself, it seems equally incapable of producing any change in

that which it has received, so that the law of inertia is at least the most simple, and the most natural that can be imagined. It is likewise confirmed by experience: in fact, we observe that motions are perpetuated on the earth, in proportion as the obstacles which oppose them are diminished; which induces us to think that if these obstacles were entirely removed, the motions would never cease. But the inertia of matter is most remarkable in the heavenly bodies, which for a great number of ages have not experienced any perceptible alteration. For these reasons, we shall consider the inertia of bodies as a law of nature; and when we observe any change in the motion of a body, we shall suppose that it arises from the action of some extraneous cause.

In uniform motions, the spaces described are proportional to the times; but the time employed in describing a given space, is longer or shorter according to the intensity of the moving force. These differences have suggested the idea of velocity, which in uniform motions is the ratio of the space to the time employed in describing it. In order to avoid the comparison of time and space which are heterogeneous quantities, we assume an interval of time, a second for example, as the unity of time, and in like manner a portion of space, such as a metre, for the unity of space. Time and space become then abstract numbers, which express how often they contain units of their species, and thus they may be compared

one with another. By this means, the velocity becomes the ratio of two abstract numbers, and its unity is the velocity of a body which describes a metre in a second. By reducing in this manner, the space, time, and velocity to abstract numbers, it appears that the space is equal to the product of the velocity into the time, which latter is consequently equal to the space divided by the velocity.

Force being known to us by the space which it causes to be described in a given time, it is natural to assume this space as its measure. But this supposes that several forces, acting in the same direction, would cause to be described in a second of time, a space equal to the sum of the spaces which each would have caused to be described separately in the same time, or in other words, that the force is proportional to the velocity; but of this we cannot be assured *a priori*, (*a*) in consequence of our ignorance of the nature of the moving force. Therefore it is necessary, on this subject, also to have recourse to experiments; for whatever is not a necessary consequence of the few data which we possess on the nature of things, must be to us the result of observation.

The force may be expressed by an infinity of functions of the velocity, which do not imply a contradiction. There is none, for instance, in supposing it proportional to the square of the velocity. In this hypothesis, it is easy to determine the motion of a point solicited by any number of

forces, the velocities of which are known; for if we assume on the directions of these forces, right lines representing their velocities, reckoning from their point of concourse, and if from the same point other lines be taken which are to each other as the squares of the first assumed lines, these lines will represent the forces themselves. By compounding them according to the rules already laid down, we shall obtain the direction of the resulting force, and also the right line which represents it; and which will be to the square of the corresponding velocity as the right line which represents one of the composing forces, to the square of its velocity. By this it appears how the motion of a point may be determined, whatever be the function of the velocity which expresses the force. Among all the functions mathematically possible, let us examine which is that of nature.

It is observed on the earth, that a body sollicited by any force whatever moves in the same manner, whatever be the angle which the direction of this force makes with the direction of the motion which is common to the body, and to the part of the terrestrial surface to which it corresponds. A slight change in this respect, would produce *(b)* a very sensible difference in the durations of the oscillations of a pendulum, according to the position of the vertical plane in which it it oscillates; but it appears from experiment, that in all vertical planes, this duration is exactly the same. In a ship which moves uniformly, a moveable body subjected to the action of a

spring, of gravity, or of any other force, moves relatively to the parts of the ship, in the same manner, whatever may be the velocity and the direction of the vessel. It may therefore be established as a general law of terrestrial motions, that if in a system of bodies which participate in a common motion, any force be impressed on one of them, its relative or apparent motion will be the same, whatever be the general motion of the system, and the angle which its direction makes that of the impressed force.

The proportionality of the force to the velocity, results from this law supposed rigorously exact; for if we suppose two bodies moving on the same right line with equal velocities, and that by impressing on one of them a force which is added to the primitive force, its velocity relatively to the other body is the same as if the two bodies had been originally in a state of rest; and it is evident that the space described by the body in consequence of the primitive force, and of that which is added to it, is then equal to the sum of the spaces which each of them would have caused it to describe in the same time, which supposes that the force is proportional to the velocity.

And conversely, if the force be proportional to the velocity, the relative motions of a system of bodies actuated by any forces whatever, are the same whatever be their common motion; for this motion being resolved into three others, parallel to three fixed axes, only increases by the same quantity, the partial velocities of each body pa-

rallel to these axes; and since the relative velocities depend only on the difference of the partial, it is the same, whatever may be the motion common to all the bodies. It is therefore impossible to judge of the absolute motion of a system, of which we make a part, by the appearances which are observed, which circumstance characterises this law, the ignorance of which has so long retarded our knowledge of the true system of the world, by the difficulty of conceiving the relative motions of projectiles above (c) the surface of the earth, which is itself carried along by a double motion, of rotation round its own axis, and of revolution about the sun.

But considering the extreme smallness of the most considerable motions which we can impress on bodies, compared with that which they have in common with the earth, it is sufficient for the appearances of a system of bodies to be independent of the direction of this motion, that a small increase in the force by which the earth is actuated mey be to (d) the corresponding increase of its velocity, in the ratio of the quantities themselves. Thus our experiments only prove the reality of this proportion, which if it really obtained, whatever the velocity of the earth might be, would give the law of the velocity proportional to the force. It would likewise give this law, if the function of the velocity which expresses it was composed of only (e) one term. If then the velocity be not proportional to the force, we must suppose that in nature the function of the velocity which expresses the force, consists of several

terms, which is very improbable. Moreover, we must suppose that the velocity of the earth is exactly such as corresponds to the preceding proportion, which is contrary to all probability. Besides, the velocity of the earth is different, in different seasons of the year; it is about one thirtieth greater in winter than in summer: this variation is still more considerable, if, as every thing appears to indicate, the solar system itself is in motion in space; for according as this progressive motion conspires with that of the earth, or is contrary to it, there should result great variations in the course of the year in the absolute motion of the earth, and this should alter the proportion of which we are speaking, and the ratio of the impressed force, to the relative velocity which results from it, if this proportion and this relation were not independent of the absolute velocity.

All the celestial phenomena serve to confirm these proofs. The velocity of light, determined by the eclipses of Jupiter's satellites, is combined with that of the earth, exactly according to the law of the proportionality of the force to the velocity; and all the motions of the solar system, computed according to this law, are entirely conformable to observations. Hence it appears that we have two laws of motion, namely, the law of inertia, and that of the force proportional to the velocity, both furnished by observation; they are the most simple and the most natural that can be imagined, and are, without doubt, derived from

the nature itself of matter; but this nature being unknown, these laws are to us only observed facts, and the only ones which the science of mechanics borrows from experience.

The velocity being proportional to the force, these two quantities may be represented the one by the other; therefore, by what goes before, we can obtain the volocity of a point solicited by any number of forces, the respective directions and velocities of which are known.

If the point is solicited by a number of forces which act in a continued manner, it will describe with a motion incessantly variable, (*f*) a curve, the nature of which will depend on the forces by which the point is solicited. To determine it, we must consider the curve in its elements, examine how they arise the one from the other, and ascend from the law of the increase of the ordinates to their finite expression. This is precisely the object of the infinitesimal calculus, the fortunate discovery of which has produced so many advantages to mechanics; hence we may perceive the utility of bringing to perfection this powerful instrument of the human mind.

We have, in the case of gravity, a daily example of a force which seems to act without intermission. It is true, we cannot determine whether its successive actions are separated by intervals of time, the duration of which is insensible, but the phenomena being nearly the same, on this hypothesis and on that of a continued action; geometricians have adopted the former, as the

most simple and commodious. Let us investigate the laws of these phenomena.

Gravity seems to act in the same manner on bodies, whether they are in a state of rest or of motion. In the first instant a body remitted to its (*g*) action, acquires an indefinitely small degree of velocity; in the second instant, an additional degree of velocity is added to the first, and so on successively; so that the velocity increases in the ratio of the times.

If we imagine a right angled triangle, one of the sides of which represents the time and increases with it, while the other side represents the velocity. The element of the area of this triangle, being equal to the product of the element of the time into the velocity, it will represent the element of the space which gravity causes a body to describe; this space will be therefore represented by the entire area of the triangle, which as it increases in the ratio of the squares of one of its sides, shews that in motion accelerated by the action of gravity, the velocities increase as the times, and the heights through which bodies fall from a state of rest, vary as the squares of the times, or of the last acquired velocities. Therefore if the space through which a body descends in the first second, be represented by unity, it will descend through four unities in two seconds, through nine unites in three seconds, and so on; so that in the successive seconds, it will describe spaces which increase as the odd numbers, 1, 3, 5, 7, &c.

The space which a body actuated by the velo-

city acquired at the end of its fall, will describe in a time equal to that of the fall, will be represented by the product of this time into its velocity; this product is double of the area of the triangle, therefore, a body moving uniformly with its last acquired velocity, will describe in a time equal to that of its fall, (*h*) a space double of that through which it has fallen.

The ratio of the last acquired velocity to the time, is constant for the same accelerating force; it increases or diminishes according as these forces are greater or less; it may therefore serve to express them. As the product of the time into the velocity is double of the space described, the accelerating force is equal to double of the space described divided by the square of the time; it is also equal to the square of the time divided by this double space. These three formulæ for expressing the accelerating forces (*i*), are useful on various occasions; they do not give the absolute values of these forces, but only their ratio to each other, which is all that is required in mechanics.

On an inclined plane, the action of gravity is decomposed into two others; the one perpendicular to the plane which is destroyed by its resistance; the other parallel to the plane, which is to the primitive force of gravity, as the height of the plane to its length. Therefore the motion on an inclined plane is uniformly accelerated; but the velocities and the spaces described, are to the velocities and spaces described in the same

time, in the direction of the vertical, as the height of the plane to its length. It follows from this, that all the chords of circles, which are (*k*) terminated in one of the extremities of the vertical diameter, are described by the action of gravity, in the same time as its diameter.

A body projected in the direction of any right line whatsoever, deviates from it continually, describing a curve concave to the horizon, of which this right line is the first tangent. Its motion when referred to this right line by vertical ordinates, is uniform, but it is accelerated in the direction of the verticals, according to the laws already explained; therefore, if from (*l*) every point of the curve verticals be extended to meet the first tangent, they will be proportional to the squares of the corresponding intercepts of this tangent, which is the characteristic property of the parabola. If the force of projection is in the direction of the vertical itself, the parabola is confounded with the vertical line, and thus the formulæ for parabolic motion give those for accelerated or retarded motions, in the direction of the vertical.

Such are the laws of the descent of heavy bodies discovered by Gallileo; at the present day, it seems to require no great power of mind to have discovered them; but since they eluded the investigations of philosophers, although perpetually presented to them by the phenomena, it

must no doubt have required an extraordinary genius to have developed them.

We have seen in the first book, that a material point suspended at the extremity of a straight line supposed without mass, and firmly fixed at its other extremity, constitutes the simple pendulum. This pendulum, when removed from its vertical position, tends to return by its gravity, and this tendency is very nearly proportional to this deviation, when it is not considerable. Suppose that two pendulums of the same length, depart at the same (*m*) instant from the vertical position, with very small velocities. In the first instant, they will describe arcs proportional to their velocities; at the commencement of the second instant, equal to the first, the velocities will be retarded proportionally to the arcs described, and consequently to the primitive velocities; therefore the arcs described in this instant will be also proportional to these velocities, and this will be likewise true for the arcs described in the third, fourth, &c. instants; thus at every instant the velocities, and the arcs measured from the vertical, will be proportional to the primitive velocities, consequently the pendulums will arrive at the state of rest, simultaneously. They will return again to the vertical with a motion accelerated, according to the same laws by which their velocites had been previously retarded, and they will reach it at the same instant, and with their primitive velocity. They will oscillate in the same manner on the other side of the vertical, and they would thus continue to oscillate for ever,

but for the resistances they meet with. It is evident that the extent of their oscillations depends on their primitive velocities, but the duration of these oscillations is the same, and consequently independent of their amplitude. The force which accelerates or retards the pendulum, is not exactly proportional to the arc measured from the vertical; so that when a body moves in a circle (*n*) the isochronism relatively to the small oscillations of a heavy body, is only approximative. But it is rigorously exact in a curve, in which the gravity resolved parallel to the tangent, is proportional to the arc reckoned from the lowest point of the curve, which immediately gives its differential equation. Huygens, to whom we are indebted for the application of the pendulum to clocks, was the first who investigated the nature of this curve. He found that it was a cycloid, the plane of which was vertical, the vertex being the lowest point; and in order that a body suspended at the extremity of an inextensible thread, should describe this curve, it was only required that the other extremity should be fixed at the point of concourse of two cycloids equal to that to be described, and situated vertically in an opposite direction, in such a manner that the thread in its vibration might envelope alternately each of these curves. But whatever ingenuity may have been displayed in these investigations, a long experience has given the preference to the circular pendulum, as being more simple, and sufficiently accurate to be applied even to the astronomical computa-

tions. But the theory of evolutes which has been suggested by them, is become very important by its applications to the system of the world.

The duration of the very small oscillations of a circular pendulum, is to the time of a body's descent through a height equal to double of the length of the pendulum, as the semi-circumference is (*o*) to the diameter. Consequently the time of descent through a small arc terminated by the vertical diameter, is to the time of descent down the diameter, or what comes to the same thing, to the time required to describe the chord of the arc, as the fourth part of the circumference to the diameter; therefore the *right* line connecting two given points, is not the line of quickest descent from the one to the other. The investigation of this line has excited the attention of geometers, and they have (*p*) found that it is a cycloid, the origin of which coincides with the most elevated point.

The length of the simple pendulum which vibrates seconds, is to twice the height through which bodies fall by the force of gravity in the same time, as the square of the diameter to the square of the (*q*) circumference. As this length may be measured with great precision, the time which heavy bodies take to descend through a determinate space may be obtained by this theorem much more accurately than by direct experiments. It has been observed in the first book, that by means of very exact experiments, the length of the pendulum vibrating seconds at Paris, has been determined to be $0^{m}741887$, hence it fol-

lows that gravity causes bodies to fall through $3^m,66107$, in the first second. This connection between the time of oscillation, the duration of which may be precisely observed, and the rectilinear motion of heavy bodies, is an ingenious remark, for which we are also indebted to Huygens.

The durations of very small oscillations of pendulums of different lengths, and actuated by the same force of gravity, vary as the square roots of these lengths. If the length of the pendulums be the same, but actuated by different forces, the times of their oscillations will be reciprocally as the square roots of the force of gravity. It is by means of these theorems that the variation of the force of gravity at the surface of the earth, and on the summit of mountains, has been determined. From observations made on pendulums, it has been likewise inferred, that gravity depends (*r*) neither on the figure nor on the surface of bodies; but that it penetrates their inmost parts, and tends to impress on them equal velocities in equal times. To be assured of this, Newton made several bodies of the same weight, but of different figures and matter, to oscillate, by placing them in the interior of the same surface, in order that they may experience the same resistance from the air. And though he instituted these experiments with the greatest accuracy possible, he was never able to perceive the smallest difference in the length of simple pendulums, vibrating seconds, as inferred from the durations of the oscillations of these bodies; hence it follows, that if bodies did not experience

any resistance in their fall, the velocity which they would acquire by the action of gravity, would be always the same in equal times.

We have likewise in circular motion, an instance of a force which acts without intermission. The motion of matter abandoned to itself being uniform and rectilineal, it is evident that a body which moves on a curve must perpetually tend to recede from the centre in the direction of the tangent. The effort which it makes for this purpose is termed *centrifugal force*; and the force directed to the centre is called a *central* or *centripetal force*. In circular motion the central force is equal and directly contrary to the centrifugal force; it tends incessantly to draw the body from (*s*) the centre to the circumference, and in an extremely short interval of time its effect may be measured by the versed sine of the small arc described.

We are enabled by this result, to compare the force of gravity with the centrifugal force which arises from the rotatory motion of the earth. At the equator, bodies describe in consequence of this rotation in each second of time, an arc of $40'',1095$ of the periphery of the terrestrial equator. As the radius of this equator is very nearly 6376606^{m}, the versed sine of this arc will be $0^{m},0126559$. The force of gravity causes bodies to descend through $3^{m},64930$ in a second at the equator; therefore the central force necessary to retain bodies at the surface of the earth, and consequently the centrifugal force arising from the rotatory motion, is to the force of gravity at the equator,

in the ratio of 1 to 288.4. As the centrifugal force acts in opposition to gravity at the equator, it diminishes it, and bodies descend to the earth by the difference only between these two forces; therefore if the entire weight which would subsist without this diminution be called *gravity*, the centrifugal force at the equator is very nearly the $\frac{1}{289}$th part of gravity. If the rotation of the earth was seventeen times more rapid, the arc described at the equator in a second, would be seventeen times greater, and its versed sine would be 289 times more considerable, consequently the centrifugal force would be equal to the force of gravity, and bodies at the equator would cease to gravitate to the earth.

In general, the expression of a constant accelerating force which acts always in the same direction, is equal to twice the space which it causes to be described, divided by the square of the time, every accelerating force, in an extremely short ininterval of time, may be considered constant and acting in the same direction; moreover, the space which the central force causes to be described in circular motion, is the versed sine of the arc described, and this versed sine is very nearly equal to the square of the arc divided by the diameter; the expression of this force is therefore the square of the arc described, divided by the square of the time, and by the radius of the circle. The arc divided by the time is the velocity itself of the body; consequently the central force,

and likewise the centrifugal force, are equal to the square of the velocity divided by the radius.

A comparison of this result, with that found above, according to which the gravity is equal to the square of the acquired velocity divided by twice the space described in the direction of the (t) vertical, shews that the centrifugal force is equal to the force of gravity, when the velocity of the revolving body is the same as that acquired by a heavy body, in falling through a height equal to half the radius of the described circumference.

The velocities of several bodies moving in circles, are equal to the circumferences which they describe divided by the times of their revolutions; the circumferences being as the radii, the squares of the velocities are as the squares of the radii divided by the squares of the times. The centrifugal forces are therefore as the radii of the circumferences divided by the squares of the times of the revolutions. It follows from this, that on the different terrestrial parallels, the centrifugal force arising from the motion of rotation of the earth, is proportional to the radii of those parallels. These beautiful theorems discovered by Huygens, conducted Newton to the general theory of curvilinear motion, and thence to the law of universal gravitation.

A body which describes any curve whatever, tends to deviate from it in the direction of the tangent: now we can easily conceive a circle to pass through two consecutive elements of the

curve; this circle is termed the osculating circle, or the circle of curvature; the body may be conceived in two consecutive instants to move on the circumference of the circle; its centrifugal force is therefore equal to the square of its velocity divided by the radius of the osculatory circle; but the magnitude and position of this circle are constantly varying.

If the curve be described by the action of a force directed to a fixed point; this force may be resolved into two, one in the direction of the radius of the osculating circle, the other in the direction of the element of the curve. The first is in equilibrio with the centrifugal force, the second increases (*u*) or diminishes the velocity of the body, therefore this velocity continually varies, but it is always such, *that the areas described by the radius vector about the origin of the force, are proportional to the times. Conversely, if the areas traced by the radius vector about a fixed point, increase proportionally to the times, the force which solicits the body, is constantly directed towards this point.* These fundamental propositions in the theory of the system of the world, are easily demonstrated in the following manner.

The accelerating force may be supposed to act only at the commencement of each instant, during which the motion of the body is uniform; the radius vector will thus describe a small triangle. If the force should cease to act in the following instant, the radius vector will trace in this second

instant a small triangle equal to the first; because the vertices of these two triangles being at the fixed point, which is the origin of the force, their bases, which exist in the same right line, will be equal; as being described with the same velocity, during two equal and consecutive instants. But at the commencement of the second instant, the accelerating force combined with the tangential force of the curve, causes the body to describe the diagonal of a parallelogram, of which the adjacent sides represent these forces. The triangle which the radius vector describes in consequence of the action of this combined force, is equal to that which would have been described without the action of the accelerating force; for these two triangles are situated on the same base, namely, the radius vector of the end of the first instant, and their vertices exist on a right line parallel to this base; therefore the areas traced by the radius vector in two consecutive instants, are equal; and consequently the sector described by this radius increases as the number of these instants, or as the times. It is evident that this only obtains when the accelerating force is directed towards the fixed point; otherwise the triangles which we have considered will not have the same altitude. Therefore, the proportionality of the areas to the times, demonstrates that the accelerating force is constantly directed to the origin of the radius vector.

In this case, if we suppose a very small sector to be described in a very short interval of time,

and if from the first extremity of the arc of this sector, a tangent to the curve be drawn, and the radius vector drawn from the origin of the force to the other extremity of the vector be prolonged to meet this tangent, it is evident that the part of this radius intercepted between the curve and the tangent, will be the space which the central force would cause the body to describe. If twice this space be divided by the square of the time, we obtain an expression for this force; but since the sector is proportional to this time, the central force is proportional to the part of the radius vector intercepted between the curve and the tangent, divided by the square of the sector. Strictly speaking, the central force in different points of the curve is not proportional to these quotients, but the accuracy is always greater according as the sectors are taken smaller, so that it is exactly proportional to the limits of these quotients. When the nature of the curve is known, this limit may be obtained in a function of the radius vector, by means of the differential analysis, and then that function of the distance to which the central (*v*) force is proportional will be determined.

If the law of the force be given, the investigation of the curve described presents greater difficulties. But whatever be the nature of the forces by which a body is actuated, the differential equations of its motion may be determined in the following manner: let us imagine three axes perpendicular to each other; the position of a body

at any instant will be determined by three coordinates parallel to these axes. Each force which acts on the point being resolved into three others parallel to the same axes, the product of the resultant of all the forces, parallel to one of the coordinates, into the element of time during which it acts, will express the increment of the velocity parallel to this coordinate; but this velocity being equal to the element of the coordinate divided by the element of the time, the differential of the quotient of this division, is equal to (*x*) the preceding product. The consideration of the two other coordinates furnishes two similar equations; thus the determination of the motion of a body, becomes an investigation of pure analysis, which is reduced to the integration of these differential equations.

In general, the element of time being supposed to be constant, the second differential of each coordinate divided by the square of this element, represents a force, which being applied to the point, in an opposite direction constitutes an equilibrium with the force which solicits it in the direction of this coordinate. If the difference of these forces be multiplied by the arbitrary variation of the coordinate, the sum of the three similar products relative to the three coordinates will be equal to cypher by the condition of equilibrium. If the point be free, the variations of the three coordinates will be all arbitrary, and by putting the coefficient of each of them respectively equal to cypher, the (*y*) three differential

equations relative to the motion of a point will be obtained. But if the point is not entirely free, there will be given one or two relations between the three coordinates, which will furnish a corresponding number of equations between their arbitrary variations. If then a like number of variations be eliminated by means of these relations, the coefficients of the remaining variations will be respectively equal to cypher; and the differential equations of motion will be obtained, which being combined with the relations existing between the coordinates, will determine the position of the point, for any instant.

The integration of these equations is easy when the force is directed to a fixed point, but very often it becomes impossible from the nature of the forces. Nevertheless the consideration of the differential equations leads to some interesting principles of mechanics, such as the following. The differential of the square of the velocity of a point subject to the action of accelerating forces, is equal to twice the sum (z) of the products of each force into the small space advanced by the body in the direction of this force; from which it is easy to infer, that the velocity acquired by a heavy body descending along a line or a curved surface, is the same as it would acquire in falling vertically through the same height.

Several Philosophers, struck with the order which prevails in nature, and with the fecundity of its means in the production of phenomena, have supposed that she always accomplishes her

ends in the simplest manner possible. In extending this conjecture to mechanics, they have investigated what was the economy of nature in the employment of forces and of time. Ptolemy ascertained that reflected light passed from one point to another, by the shortest possible route, and consequently in the least time, the velocity of the luminous ray being supposed to be always the same. Fermat, one of the most original men that France ever produced, generalized this principle, by extending it to the refraction of light. He supposed therefore that it passes from a point assumed without a diaphonous medium to an interior point, in the shortest possible time; then supposing that the velocity is less in this medium than in a vacuo, which is extremely probable, he investigated the law of the refraction of light in these hypotheses. By applying to this problem his beautiful method *de maximis* and *de minimis*, (which should be considered as the true origin of the differential calculus,) he found agreeably to experience, that the sines of (*aa*) incidence and of refraction ought to be in a constant ratio, greater than unity. The ingenious manner in which Newton deduced this ratio from the attraction of the media which the rays traverse, indicated to Maupertius, that the velocity of light increases in diaphanous media, and that consequently it is not, as Fermat supposed, the sum of the quotients of the spaces described in a vacuo and in the medium, divided by their corresponding velocities, but the sum of the products of these quantities which should be a *mini-*

mum. Euler extended this hypothesis to motions which are every moment variable, and he demonstrated by several examples, that of all the curves that a body may describe in passing from one point to another, it always selects that *in which the integral of the product of its mass, into its velocity and the element of the curve, is a minimum.* Thus the velocity of a point which moves on a curved surface, and is not actuated by any force, being constant, it passes from one point to another by the shortest line (*bb*) which can be traced on this surface. The preceding integral has been termed *the action of the body*, and the sum of all the similar integrals relative to each body of the system, has been called the action of the system. Therefore Euler has demonstrated that this action is a *minimum,* so that the economy of nature consists in sparing this action; this is what constitutes *the principle of least action,* the discovery of which is certainly due to Euler; though Lagrange has since derived it from the primordial laws of motion. But this principle is only at bottom a remarkable result of those laws, which are, as we have seen, the most simple and the most natural that can be conceived, and which seem to be derived from the very essence of matter. All laws mathematically possible between the force and the velocity, furnish analogous results, provided that we substitute in this principle, instead of the velocity, that function of the velocity by which the force is expressed. There-

fore the principle of the least action ought not to be elevated to the rank of a final cause, for so far from having given birth to the laws of motion, it has not even contributed to their discovery, without which we would still dispute about what was to be understood by the last action of nature.

CHAP. III.

Of the equilibrium of a system of bodies.

THE simplest case of equilibrium between several bodies, is that of two material points meeting each other, with equal and directly contrary velocities. Their mutual impenetrability, that property of matter which prevents two bodies from occupying the same place at the same instant, evidently annihilates their velocities, and reduces them to a state of rest. But if two bodies of different masses impinge on each other, with opposite velocities, what relation exists between the velocities and the masses in the case of an equilibrium? In order to solve this problem, suppose a system of contiguous material points arranged in the same right line, and actuated by a common velocity, in the direction of this line; suppose also a second system of contiguous material points, situated on this same line and actuated also by a common velocity, but in a direction opposite to the preceding, so that the two systems, after impinging on each other, may constitute an equilibrium. It is evident, that if the first system consisted of only one material point, each point in

the second system would destroy in the striking point, a part of its velocity equal to the velocity of the second system; therefore in the case of equilibrium, the velocity of the striking point should be equal to the product of the velocity of the second system into the number of material points composing it, and thus we may substitute for the first system, one sole point actuated by a velocity equal to this product. We may likewise substitute in place of the second system, a material point actuated by a velocity, equal to the product of the velocity of the first system, into the number of its material points. Thus in place of the two systems we shall have two points which will sustain each other in equilibrio with contrary velocities, of which one will be the product of the velocity of the first system into the number of its points, and of which the other will be the product arising from multiplying the velocity of the points of the second system by their number; therefore in the case of an equilibrium these products should be equal to each other. The mass of a body is the aggregate of its material points. The product of the mass by the velocity is termed *the quantity of motion*; this is also what is understood by *the force of a body*. In order that two bodies, or two systems of points, which impinge on each other in opposite directions, may be in equilibrio, the quantities of motion or the opposite forces should be equal, and consequently the velocities should be inversely as the masses.

Two material points cannot act, the one on the

other, except in the direction of the right line which connects them : the action which the first exercises on the second communicates to it a certain quantity of motion ; now we may conceive that previous to the action, the second body is actuated by this quantity, and by another which is equal and directly opposite to it, the action of the first body is therefore employed (*a*) in destroying this last quantity of motion, but to effect this, it must employ a quantity of motion equal and contrary to that which is to be destroyed. Hence it appears generally, that in the mutual action of bodies, the reaction is always equal and directly contrary to the action. It likewise appears, that this equality does not imply the existence of any particular force inherent in matter, but results from this, that a body cannot acquire motion from the action of another, without depriving it of a portion of its motion ; in the same manner, as a vessel can only be filled at the expence of another which communicates with it.

The equality between action and reaction manifests itself in all the actions of nature ; iron attracts the magnet as it is attracted by it ; the same is observed in electric attractions and repulsions, and even in the developement of animal forces ; for whatever be the nature of the prime motive power in man and animals, it is clear that they experience, from the reaction of matter, a force equal and contrary to that which they communicate to it, and that consequently when they are

considered in this point of view, they are subject to the same laws as inanimate beings.

The reciprocity of the velocity to the mass in the case of equilibrium, enables us to determine the ratio of the masses of different bodies. Those of homogeneous bodies are proportional to their volumes, which geometry teaches us to measure; but all bodies are not homogeneous, and from the differences which exist either in their integrant molecules or in the number and magnitude of the intervals or pores which separate those molecules, there arise very considerable diversities in the masses which are contained under the same volume. Geometry then becomes inadequate to determine these masses, and we are necessarily obliged to have recourse to mechanics.

If we conceive that in two globes composed of different substances, their diameters are so varied, that they may constitute an equilibrium when they meet with equal and directly opposite velocities, we may be assured that then they contain the same number of material points, and that consequently their masses are equal. The ratio of the volumes of these substances, the masses being equal, will thus be obtained; and afterwards, we can determine by geometry, the ratio of the masses of any two volumes of the same substance. But this method would be extremely troublesome in the numerous comparisons which are continually required in the various relations of commerce. Fortunately, nature furnishes, in the

weight of bodies, a simple method of comparing their masses.

It has been observed in the preceding chapter, that every material point in the same place on the earth, tends to move with the same velocity by the action of gravity. The sum of these tendencies is that which constitutes the weight of a body; therefore *(b)* the weights are proportional to the masses. It follows from this, that if two bodies suspended at the extremities of a thread, which passes over a pully, are in equilibrio when an equal portion of the thread is on each side of the pully, the masses of those bodies are equal, because tending to move with the same velocity by the action of gravity, their mutual action on each other is precisely the same, as if they impinged on each other, with equal and directly contrary velocities. Likewise if two bodies placed in a balance, of which the arms and plates are perfectly equal, be in equilibrio, we may be assured of the equality of their masses. The ratio between the masses of different bodies may thus be obtained by means of an exact and sensible balance, and of a great number of small equal weights, by determining how many of these weights are necessary to retain these masses in equilibrio.

The density of a body depends on the number of its material points, included in a given volume; it is therefore proportional to the ratio of the mass to the volume.

The density of a substance destitute of pores

would be the greatest possible; and a comparison of its density with that of other bodies, would give the quantity (*c*) of matter which they contain. But as we are not acquainted with any such substance, we can only obtain the relative densities of bodies; these densities are in the proportion of the weights when the volumes are the same, for the weights are proportional to the masses: assuming therefore as unity, the density of any substance, at a constant temperature, for instance, the *maximum* of the density of distilled water, the density of a body will be the ratio of its weight to that of an equal volume of water reduced to its maximum density. This ratio is termed its *specific gravity*.

What has been said seems to suppose that matter is homogeneous, and that bodies only differ from each other in the figure and magnitude of their pores and of their integrant molecules. It is however possible that there may be essential differences in the very nature of these molecules; and it is not repugnant to the limited information which we possess of matter, to suppose the celestial regions filled with a fluid devoid of pores, and still of such a nature as not to oppose any sensible resistance to the planetary motions; we may thus reconcile the uniformity of these motions, which is (*d*) evinced by the phenomena, with the opinion of those philosophers who regard a vacuum as an impossibility; but this is of no consequence in mechanics, which takes into account no other properties of matter, but extension and mo-

tion. We may therefore, without any apprehension of error, assume the homogeneity of the elements of matter, provided that by equal masses we understand masses which being sollicited by equal and directly contrary velocities, constitute an equilibrium.

In the theory of the equilibrium, and motion of bodies, we abstract from the consideration of the number and figure of the pores which are distributed through them. But we may have regard to the differences of their respective densities, by supposing them to be constituted of material points more or less dense, which in fluids are perfectly free, and which in hard bodies are connected by inflexible straight lines, destitute of mass, and which in elastic and soft bodies, are connected by flexible and extensible lines. It is evident that in these hypotheses, bodies should present the appearances which they actually exhibit.

The conditions of the equilibrium of a system of bodies may be always determined by the law of the composition of forces, which has been explained in the first chapter of this book; for we may conceive the force by which every material point of the system is actuated, to be applied to that point of its direction where all the forces which destroy it concur, or which by combining with it, constitute a resultant, which in the case of equilibrium is destroyed by the fixed points of the system. Let us consider, for example, two material points, attached to the extremities of an inflexible lever, and suppose that the forces which

sollicit them exist in the plane of the lever: these forces being supposed to meet at the point of concourse of their directions, their resultant should, in order to constitute an equilibrium, pass through the fulcrum, which can alone destroy it; (*e*) and according to the law of the composition of forces, the two composing forces should be reciprocally proportional to perpendiculars demitted from the fulcrum or point of support, on their directions.

If we suppose two heavy bodies to be attached to the extremities of a rectilinear inflexible lever, of which the mass is indefinitely small, relatively to the masses of these bodies, the directions respectively parallel to that of the gravity, may be conceived to meet at an infinite distance. In this case, the forces by which each body is actuated, or what is the same thing, their weights must be in the case of equilibrium reciprocally proportional to perpendiculars let fall from the fulcrum on the directions of these forces; these perpendiculars are proportional to the arms of the levers, consequently the weights of two bodies are, in the case of equilibrium, reciprocally proportional to the arms of the lever to which they are attached.

A very small weight may therefore sustain a very considerable one in equilibrio, and in this manner we can raise an enormous weight by a very slight effort; but for this purpose the arm of the lever to which the power is attached, must be very long with respect to that which elevates the weight, so that the power must describe a great space to elevate the weight a small height,

Consequently what is gained in force, is lost in time, and this is the case universally in all (*f*) machines. But we may almost always dispose of time at pleasure, when we can only employ a very limited force. In other cases where it is required to produce a great velocity, it may be effected by applying the force to the shorter arm of the lever. It is in this possibility of augmenting, according to circumstances, the mass or the velocity of the bodies to be moved, that the principal advantage of machinery consists.

From a consideration of the lever has been suggested the notion of moments. By the *moment* of a force to make a system turn about a point, is understood the product of this force, into the distance of the point from its direction. Therefore in the case of the equilibrium of a lever, to the extremities of which two forces are applied, the moments of these forces with respect (*g*) to the fulcrum or point on which it turns, must be equal and contrary, or what comes to the same thing, the sum of the moments relatively to this point must be equal to cypher.

The projection of a force on a plane drawn through a fixed point, multiplied into the distance of the point from this projection, is termed the moment of the force to make the system to revolve about an axis which passes through the fixed point, and is perpendicular to the plane.

The moment of the resultant of any number of forces with respect to a point, or any axis, is

equal to the sum of the corresponding moments of the composing forces.

Parallel forces may be supposed to meet at an infinite distance, they are therefore reducible to an unique force, equal and parallel to their sum; therefore if each force be resolved into two, one of which exists on a given plane, the other being perpendicular to this plane, all the forces situated in the plane are reducible to a unique force, as likewise all the forces which are perpendicular to this plane. There exists always a plane passing through the fixed point, such that the resultant of the forces which are perpendicular to it, either vanishes or passes through this point: in these two cases the (h) moment of this resultant vanishes relatively to the axes which have this point for the origin, and the moment of the forces of the system, with respect to these axes is reduced to the moment of the resultant situated in the plane in question. The axis about which this moment is a *maximum*, is that which is perpendicular to this plane, and the moment of the forces relative to an axis, which passing through the fixed point makes any angle with the axis of greatest moment, is equal to the greatest moment of the system, multiplied into the cosine of this angle; so that this moment vanishes for all axes situated in the plane to which the axis of the greatest moment is perpendicular.

The sum of the squares of the cosines of the angles made by the axis of greatest moment, with any three axes perpendicular to each (i) other

and passing through the fixed point being equal to unity; the squares of the three sums of the moments of the forces, with respect to these axes, are equal to the square of the greatest moment.

In order that a system of bodies connected in an invariable manner, and which revolves about a fixed point, may be in equilibrio, the sum of the moments of the forces must vanish with respect to any axis passing through this point. It follows from what goes before, that this will always be the case if the preceding sum be equal to cypher, relatively to three fixed axes, perpendicular to each other. If there is no fixed point in the system, it is required in addition to the preceding conditions, in order to insure an equilibrium, that the three sums of forces resolved parallel to these axes, be *respectively* (*k*) equal to cypher.

Let us considsr a system of ponderable points firmly connected, referred to three planes at right angles to each other, and connected with the system. The action of gravity being resolved parallel to the intersections of these planes, all the forces parallel to the same plane may be reduced to an unique resultant parallel to this plane and equal to their sum. The three resultants relative to the three planes must concur in the same point; for the action of gravity on the several points of the system being parallel, they have an unique resultant, which is obtained by first combining two of these forces, and afterwards their resultant with the third force; the resultant of the three forces with a fourth, and

so on. The situation of this point of concourse with respect to the system, is independent of the inclination of the planes to the direction of gravity; for a greater or less inclination can only change the (l) values of the three partial resultants, without altering their position with respect to the planes; therefore this point being supposed fixed, all the efforts of the weights of the system will be annihilated in all the positions which it can assume in revolving about this point, which for this reason has been termed *the centre of gravitg of the system:* Let us conceive the position of this centre, and that of the different points of the system to be determined by coordinates parallel to three axes at right angles to each other. The actions of gravity being equal and parallel, and the resultant of those actions passing in all positions of the system through its centre of gravity; if this resultant be supposed to be successively parallel to each of the three axes, the equality of the moment of the resultant to the sum of the moments of the composing forces gives any one of these coordinates, multiplied by the entire mass of the system, equal to the sum of the products of the mass of each point into its corresponding coordinate. Thus the determination of this centre, of which gravity first suggested the idea, is independent of it. The consideration of this centre extended to a system of bodies ponderable or not, free or connected in any manner whatever, is extremely useful in mechanics.

The theorem which was given at the close of the first chapter on the equilibrium of a point, when generalized, leads to the following theorem, which contains, in the most general manner, the conditions of the equilibrium of a system of material points actuated by any forces whatever.

If an indefinitely *(m)* small change be made in the position of the system, in a manner compatible with the connection of its parts, each material point will advance in the direction of the force which sollicits it, by a quantity equal to the part of this direction, comprised between the first position of the point and the perpendicular let fall from the second position of the point on this direction; this being premised, in the case of equilibrium, *the sum of the products of each force into the quantity by which the point to which it is applied advances in its direction, is equal to cypher; and conversely if this sum is equal to cypher, whatever may be the variation of the system, it is in equilibrio.* It is in this that the principle of virtual velocity consists, for which we indebted to John Bernoulli, but in applying it, it should be observed, that those products must be taken negatively, of which the points in the change of position of the system, advance in a direction contrary to that of their forces: it should be likewise recollected, that the force is the product of the mass of a material point, into the velocity with which it would move, if entirely free.

If we conceive the position of each point of the system, to be determined by three rectangular co-

ordinates, the sum of the products of each force into the quantity advanced in its direction by the point which it sollicits, when an indefinitely small change is made in the system, will be expressed by a linear function of the variation of the coordinates of its several points; these variations have with each other relations, which depend on the manner in which the parts of the system are connected together, therefore in reducing the arbitrary variations to the least possible number by means of these relations in the preceding sum which should be equal to cypher, in the case of equilibrium; it is necessary, in order that the equilibrium may take place in every direction, to make the coefficient of each of the remaining variations separately equal to cypher, which will furnish us with as many equations as there are arbitrary variations. These equations, combined (*n*) with those which are furnished by the connection of the parts of the system, will contain all the conditions of its equilibrium.

There are two states of equilibrium, which are essentially different. In one, if the equilibrium be a little deranged, all the bodies of the system only make small oscillations about their primitive position; and then the equilibrium is *firm* or *stable*. This stability is absolute, if it obtains whatever may be the oscillations of the system; it is only relative, if it only obtains with respect to oscillations of a certain species. In the other state of equilibrium, when the system is disturbed,

the bodies deviate more and more from their primitive position. We may form a just notion of these two states, by considering an ellipse situated vertically on a horizontal plane. If the ellipse be in equilibrio on its lesser axis, it is clear that by making it to deviate a little from this situation by a slight *(o)* motion on itself, it tends to revert, making oscillations which will be soon annihilated by the friction and resistance of the air. But if the ellipse be in equilibrio on its greater axis; when it once deviates from this situation, it continually deflects from it more and more, and is at length upset on its lesser axis. Consequently the stability of the equilibrium depends on the nature of the small oscillations, which the system, when deranged in any manner, makes about this state. In order to determine generally in what manner the different states of stable and tottering equilibrium succeed each other, let us consider a curve returning into itself, situated vertically in a position of stable equilibrium. When it is a little deranged from this state, it tends to revert to it; this tendency varies as the deviation increases, and when it vanishes, the curve is found in a new position of equilibrium, but which is not stable, for the curve previous to its arrival tended to revert to its primitive position. Beyond this last position, the tendency to the first state, and consequently to the second, becomes negative, until it vanishes a second time, and then the curve is in a position of stable equilibrium. By pursuing this

illustration, it appears that the states of stable and tottering equilibrium succeed each other alternately, like the maxima and minima of the ordinates of curves. The same reasoning may be easily extended to the different states of equilibrium of a system of bodies.

CHAP. IV.

Of the equilibrium of fluids.

THE characteristic property of fluids, whether elastic or incompressible, is the extreme facility with which each of their molecules yields to the slightest pressure which it experiences on one side, rather than on the other. We proceed therefore to establish on this property, the laws of the equilibrium of fluids, by considering them as constituted of molecules perfectly moveable among each other.

It follows immediately from this mobility, that the force by which a molecule of the free surface of a fluid, is actuated, must be perpendicular to this surface, for if it was inclined to it, by resolving the force into two others, one perpendicular, and the other parallel to this surface, the molecule would glide on the surface (*a*) in consequence of this last force. Gravity is consequently perpendicular to the surface of stagnant waters, which is on this account horizontal; for the same reason, the pressure which each fluid molecule exerts against a surface, must be perpendicular to it.

Each molecule in the interior of a fluid mass,

experiences a pressure, which in the atmosphere is measured by the height of the barometer, and which may be estimated in a similar manner for every other fluid. By considering each molecule as an (b) indefinitely small rectangular prism, the pressure of the ambient fluid will be perpendicular to the faces of this prism, which will consequently tend to move perpendicularly to each face, by virtue of the difference of pressures, which the fluid exerts on two opposite faces. From these different pressures arise three forces perpendicular to each other, which must be combined with the other forces which sollicit the molecule. It is easy to shew from this, that in the state of equilibrium the differential of the pressure is equal to the density (e) of the fluid molecule multiplied into the sum of the products of each force by the element of its direction; therefore if the fluid be incompressible and homogeneous, this sum will be an exact differential, this important result was first announced by Clairaut, in his beautiful treatise on the figure of the earth.

When the forces arise from attractions, which are always a function of the distance from the attracting centres, the product of each force into the element of its direction is an exact differential; therefore the density of the fluid molecule must be a function of the pressure, for the differential of the (d) pressure divided by this density is equal to an exact differential. Consequently all the strata of the fluid mass, in which the pressure is constant, are of the same density throughout

their entire extent. The resultant of all the forces which actuate each molecule at the surface of these strata, is perpendicular to this surface, on which the molecule would glide if this resultant was inclined to it. In consequence of this property these strata have been termed *strata of level.*

The density of a molecule of atmospheric air, is a function of the pressure and of the temperature; its gravity is very nearly a function of its height above the surface of the earth. If its temperature was likewise a function of this height, the equation of the equilibrium of the atmosphere, would be a differential equation between the pressure and the elevation, and consequently the equilibrium (*d*) would be always possible. But in nature, the temperature of the different regions of the atmosphere depends also on the latitude, on the presence of the sun, and on a thousand variable or constant causes which ought to produce in this great fluid mass, motions often very considerable. In consequence of the mobility of its molecules, a heavy fluid may produce a pressure much more considerable than its weight. For example, a small column of water, terminated by a large horizontal surface, presses the base on which it is incumbent, as much as a cylinder of water of the same base and height. In order to evince the truth of this paradox, suppose a fixed cylindrical (*e*) vase, of which the horizontal base is moveable; and let this vase be filled with water, its base is sustained in equilibrio

by a force equal and contrary to the pressure which it experiences. It is evident that the equilibrium would still obtain, in the case in which a part of the water was to consolidate and unite itself with the sides of the vessel; for the equilibrium of a system of bodies, is not deranged by supposing that in this state, several of them unite or become attached to fixed points. We may in this manner form an infinity of vessels of different figures, having all the same height and base as the cylindrical vessel, and in which the water will exert the same pressure on the moveable base.

In general, when a fluid acts only by its weight, the pressure which it exerts against a surface, is equivalent to the weight of a prism of this fluid, of which the base is equal to the pressed surface, (*f*) and of which the height is equal to the distance of the centre of gravity of this surface, from the plane of the level of the fluid.

A body plunged in a fluid, loses a part of its weight equal to the weight of a volume of the displaced fluid; for before the immersion, the surrounding fluid was in equilibrio with the weight of this volume of the fluid, which may be supposed, without deranging the equilibrium, to have formed itself into a solid mass, the resulting force of all the actions of the fluid on this mass must therefore be in equilibrio with its weight, and pass through its centre of gravity; now it is clear that (*g*) the same actions are exerted on a body which occupies its place; consequently the action of the

fluid destroys a part of the weight of this body, equal to the weight of the volume of the displaced fluid. Hence it follows that bodies weigh less in air than in a vacuo; the difference, though for the most part hardly perceptible, should not be neglected in very delicate experiments.

By means of a balance, which carries at the extremity of one of its arms a body which can be plunged in a fluid, we can estimate exactly the diminution of weight which the body experiences in this immersion, and determine its *specific gravity*, or its density relative to that of fluid. This gravity is the ratio of the weight of the body in a vacuo, to its loss of weight, when it is entirely immersed in the fluid. It is thus that the specific gravities of bodies have been determined, by comparing them with distilled water at its *maximum* density.

In order that a body which is lighter than a fluid may be in equilibrio at its surface, its weight must be equal to the volume of the displaced fluid. It is moreover necessary, that the centres of gravity of this portion of the fluid and of the body should exist in the same vertical line; for the resultant of the actions of gravity on all the molecules of the body, passes through its centre of gravity, and the resultant of all the actions of the fluid on this body passes (*h*) through the centre of gravity of the volume of the displaced fluid; and as these resultants must be on the same right line in order to destroy each others effect, the centres of gravity must exist in the same vertical.

But in order to secure the *stability* of the equilibrium, it is necessary that other conditions, besides the two preceding, should be satisfied. It may be always determined by the following rule.

If through the centre of gravity of the section of a floating body on a level with the water, we conceive a horizontal axis, such that the sum of the products of each element of the section, into the square of its distance from this axis be less than a similar sum relatively to any other horizontal axis drawn through the same centre, the equilibrium will be stable in every direction, when this sum is greater than the product of the volume of the displaced fluid, into (*i*) the height of the centre of gravity of the body, above the centre of gravity of this volume. This rule is principally useful, in the construction of vessels to which it is necessary to give sufficient stability, in order to enable them to resist the efforts of storms and waters which tend to submerge them. In a ship the axis drawn from the stern to the prow is the line, relatively to which, the above mentioned sum is a *minimum*; it is therefore easy by means of the preceding rule, to determine the stability.

Two fluids contained in a vessel, dispose themselves in such a manner that the heaviest occupies (*k*) the lowest part of the vessel, and the surface which separates them is horizontal.

If two fluids communicate with each other by means of a bent tube, the surface which separates them in a state of equilibrium is nearly horizontal, when the tube is very large; their heights

above this surface, are reciprocally proportional to their specific gravities. The entire atmosphere being therefore supposed to be of a uniform density, equal to that of the air at the temperature of melting ice; its height will be 7963^{m}, when compressed by a column of mercury of seventy-six centimetres; but because the density of the atmospheric strata diminishes, as they are more elevated above the level of the seas, the height of the atmosphere is much greater.

CHAP. V.

Of the motion of a system of bodies.

Let us consider first, the action of two material points of different masses, which moving in the same right line impinge on each other. We may conceive immediately before to the shock, their motions to be decomposed in such a manner, that they may have a common velocity, and two opposite velocities, such that if they were actuated by these alone they would have remained in equilibrio. The velocity common to the two points, is not affected by their mutual action, and therefore it will subsist alone after the shock. To determine it we shall observe, that the quantity of motion of the two points arising from this common velocity, plus the sum of the quantities of motion which are due to the velocities, which are destroyed, represent the sum of the quantities of motion previous to the shock, provided that the quantities of motion arising from the opposite velocities, be taken with contrary signs; but (*a*) by the conditions of equilibrium, the sum of the quantities of motion produced by the destroyed velocity vanishes; hence, the quantity of motion arising from the common velocity, is equal to that which existed in the two points previous

to the impact; and consequently this velocity is equal to the sum of the quantities of motion, divided by the sum of the masses.

The impact of two material points is purely ideal, but it is easy to reduce to it that of any two bodies, by observing that if these bodies impinge in the direction of a right line passing through their centres of gravity, and perpendicular to their surfaces of contact, they will act on each other as if their masses were condensed into these centres; therefore motion is communicated between them, as between two material points, of which the masses are respectively equal to these bodies.

The preceding demonstration supposes, that after the shock, the two bodies must have the same velocity. We may readily suppose that this must be the case for soft bodies, in which the communication of motion is made successively, and by insensible gradations; for it is evident, that from the instant when the struck body has the same velocity as the striking body, all velocity between them ceases. But between two bodies of absolute hardness, the shock is instantaneous, and it does not appear to be necessary that their velocities should be (*b*) afterwards the same; their mutual impenetrability solely requires that the velocity of the striking body should be less; in other respects it is indeterminate. This indetermination demonstrates the absurdity of an absolute hardness. In fact, in nature the hardest bodies, if they are not elastic, have an

imperceptible softness, which renders their mutual action successive, although its duration is insensible.

Where bodies are perfectly elastic, it is necessary, in order to obtain their velocity after the shock, to add or subtract from the common velocity which they would have, if they were destitute of elasticity, the velocity which they would gain or lose in this hypothesis; for the perfect elasticity doubles these effects, by the restitution of the springs which were compressed by the shock; therefore the velocity of each body after the shock will be obtained by subtracting its velocity before the shock, from twice this common velocity.

Hence it is easy to infer, that the sum of the products of each mass by the square of its velocity, is the same before and after the shock of the two bodies; which obtains universally in the impact of any number of perfectly elastic bodies, however they may be supposed to act on each other.

Such are the laws of the communication of motion by impulse, laws which have been confirmed by experience, and which may be mathematically deduced from the two fundamental laws of motion, explained in the second chapter of this book. Several philosophers have endeavoured to determine them from the consideration of final causes. Descartes, supposing that the quantity of motion in the universe should always remain the same without any regard to its direction, has deduced from this false hypothesis erroneous

laws of the communication of motion, which furnish a remarkable example of the errors to which we are liable, when we endeavour to develope the laws of nature, by attributing to her, particular views.

When a body receives an impulsion, in a direction which passes through its centre of gravity, all its parts move with an equal velocity. If this direction is at one side of this point, the velocities of different parts of this body will be unequal, and from this inequality arises a motion of rotation of the body about its centre of gravity, at the same time that this centre is carried forward with the velocity with which it would have moved if the direction of the impulsion had passed through this point. This (*c*) case is that of the earth, and of the planets. Thus to explain the double motion of rotation and of translation of the earth, it is sufficient to suppose that in the beginning, it received an impulse of which the direction was at a small distance from its centre of gravity, and supposing this planet to be homogeneous, this distance is very nearly the hundreth and sixtieth part of its radius. It is extremely improbable that the primitive direction of the planets, the satellites and comets, should pass exactly through their centres of gravity; all these bodies should therefore revolve round their axes.

For the same reason the sun, which revolves on an axis, must have received an impulsion, of which the direction not passing accurately through its centre of gravity, carries it along in

space with the planetary system, unless an impulse in a contrary direction should have destroyed this motion, which (*d*) is not at all probable.

The impulsion given to an homogeneous sphere, in a direction which does not pass through its centre, causes it to revolve constantly round a diameter perpendicular to a plane passing through its centre, and through the direction of the impressed force. New forces which sollicit all its points, and of which the resulting force passes through its centre, do not alter the parallelism of the axis of rotation. It is thus that we explain how the axis of the earth, remains always very nearly parallel to itself in its revolution about the sun, without assuming with Copernicus, an annual motion of the poles of the earth about those of the ecliptic. If the body be of any figure whatever, its axis of rotation may vary at every instant: the investigation of these variations, whatever be the forces which act on the body, is one of the most interesting problems in the science of mechanics which relates to hard bodies, in consequence of its connexion with the procession of the equinoxes and the libration of the moon. Its solution has led to this curious and useful result, namely, that in every body there exist three axes, perpendicular to each other, about which it may revolve (*e*) uniformly, when it is not sollicited by any external force. These axes have on this account been termed the *principal axes of rotation*. They possess this remarkable property, that the sum of

the products of each molecule of the body, into the square of its distance from the axis, is a *maximum* with respect to two of these axis, and a *minimum* with respect to the third. If we suppose the body to revolve round an axis which is inclined in a very small angle to either of the two first, the instantaneous axis of rotation will always deviate from either of them by an indefinitely small quantity; therefore the rotation is stable relatively to the two first axes; it is not so with respect to the third principal axis, and if the instantaneous axis deviates from it, by ever so small (*f*) a quantity, this deviation will increase and become continually greater and greater.

A body, or a system of bodies of any figure whatever, oscillating about a fixed horizontal axis, constitutes the compound pendulum. These are the only species of pendulums which really exist in nature, and the simple pendulums, which have been noticed in the second chapter, are purely geometrical conceptions which have been (*g*) devised in order to simplify the subject. It is easy to reduce to them the compound pendulums, of which all the points are firmly connected together. If the length of the simple pendulum, the oscillations of which are of the same duration as those of the compound pendulum, be multiplied by the mass of this last pendulum, and by the distance of its centre of gravity from the axis of oscillation, the product will be equal to the sum of the products of each molecule of the compound pendulum, into the square of its distance from the same axis. It is by means of

this rule, which was discovered by Huygens, that experiments on compound pendulums make known the length of the simple pendulum which vibrates seconds.

Conceive a pendulum to make very small oscillations, all of which exist in the same plane, and suppose that at the moment of its greatest deviation from the vertical, a small force is impressed on it, perpendicular to the plane of its motion; it will describe an ellipse about the vertical. In order to represent this motion, we may conceive a fictitious pendulum which continues to vibrate as the real pendulum would do, if the new force had not been impressed on it; while the real pendulum, in virtue of the impressed force vibrates at each side of the ideal pendulum, as if this fictitious pendulum had been immoveable and vertical. Thus it appears, that the motion of the real pendulum (*h*) is the result of two simple oscillations co-existing and perpendicular to each other.

This manner of considering the small oscillations of bodies, may be extended to any system whatever. If we suppose the system to be deranged from its state of equilibrium by very small impulsions, and that afterwards new ones are impressed on it, it will oscillate relatively to the successive states which it would have assumed in virtue of the first impulsions, in the same manner as would vibrate with respect to its state of equilibrium, if the new impulsions had been solely impressed in this state. Therefore the very small

oscillations of a system of bodies, however complicated, may be considered as made up of simple oscillations, perfectly similar to those of the (*i*) pendulum. In fact, if we conceive the system to be primitively in repose, and then very little disturbed from its state of equilibrium, so that the force which sollicits each body may tend to reduce it to this state, and may moreover be proportional to the distance of the body from this point, it is evident that this will be the case during the oscillation of the system, and that at each instant the velocity of the different points will be proportional to their distance from the position of equilibrium. They will therefore attain this position at the same instant, and they will vibrate in the same manner as the simple pendulum. But the state of derangement which we have assigned to the system, is not unique. If one of the bodies be elongated from the position of equilibrium, and if then the situations of the other bodies which satisfy the preceding conditions be investigated, we arrive at an equation of a degree equal to the number of the bodies of the system, which are moveable between themselves; which furnishes for each body, as many species of simple oscillations, as there are bodies. Let us conceive that the first species of oscillations exists in the system; and at any given instant, let all the bodies be supposed to be elongated from their position, proportionally to the quantities which are relative to the second species of oscillations. In virtue of the coexistence of the oscillations, the

system will oscillate with respect to the consecutive states, which it would have assumed in conquence of the first species of oscillation, as it would have oscillated about its state of equilibrium, if the second species had been solely impressed on it; its motion will therefore be made up of the two first species of oscillation: we may in like manner combine with this motion, the third species of oscillations, and so by proceeding in this manner combine all these species in the most general manner; we can thus synthetically compound all possible motions, which may be impressed on a system, provided that they be very small, and conversely we may by analysing these motions, resolve them into simple oscillations. Hence arises an easy method of recognizing the absolute stability of the equilibrium of a system of bodies.

If in all positions relative to each species of oscillations, the forces tend to reduce the bodies to a state of equilibrium, this state will be stable; this will not be the case, or the stability will be only relative, if in any one (k) of these positions, the forces tend to encrease the distance of the bodies from the position of equilibrium.

It is evident that this manner of viewing the very small oscillations of a system of bodies, may be extended to fluids themselves, of which the oscillations are the result of simple oscillations existing simultaneously, and frequently of an infinite number.

We have a very sensible example of the existence of very small oscillations, in the case of

waves, when a point of the surface of stagnant water is slightly agitated; circular waves are observed to form and to extend themselves about it. If the surface be agitated at a second point, new waves are observed to arise, and mix themselves with the former; they are superimposed over the surface agitated by the first waves, as they would be (*l*) dispersed on this surface, if it had remained tranquil, so that they are perfectly distinct in their commingling. What is observed by the eye to be the case with respect to waves, the ear perceives with respect to sounds or the vibrations of the air, which are propagated simultaneously without any alteration, and make very distinct impressions.

The principle of the coexistence of simple oscillations, for which we are indebted to Daniel Bernoulli, is one of these general results which assists the imagination, by the facility with which it enables us to exhibit phenomena and their successive changes.

It may be easily deduced from the analytical theory of the small oscillations of a system of bodies. These oscillations depend on linear differential equations, of which the complete integrals, are the sum of the (*m*) particular integrals. Thus the simple oscillations are disposed one on the other, to form the motion of the system, as the particular integrals which represent them, are combined together to constitute the complete integrals. It is interesting to trace in this manner, the intellectual truths of analysis in the pheno-

mena of nature. This correspondence, of which the system of the world furnishes us with numerous examples, constitutes one of the great charms of mathematical speculations.

It is natural to reduce the laws of the motion of bodies to a general principle, in the same manner as the laws of their equilibrium have been reduced to the sole principle of virtual velocities. To effect this, let us consider the motions of a system of bodies acting the one on the other, without being sollicited by accelerating forces. Their velocities change at every instant, but we may conceive each velocity at any instant to be compounded of the velocity which it would have at the following instant, and of another velocity which ought to be destroyed at the commencement of this new instant. If the velocity destroyed be known, it would be easy, by the law of the resolution of forces, to determine the velocity of the body at the second instant; now it is evident, that if the bodies were only actuated by the velocities which are destroyed, they would mutually constitute an equilibrium; thus the laws of equilibrium will give the relations of the velocities which are destroyed, and it will be easy to determine from thence the velocities which remain, and their (*n*) directions. Therefore by means of the infinitismal analysis we shall have the successive variations of the motion of the system, and its position at every instant. It is evident that if the bodies are actuated by accelerating forces, the same resolution of velocities

may be employed, but then, the equilibrium ought to obtain between the velocities destroyed and these forces.

This method of reducing the laws of motion to those of equilibrium, for which we are principally indebted to d'Alembert, is very luminous and universally applicable. It would be a matter of surprise that it had escaped the notice of geometers, who had occupied themselves with the principles of dynamics previously to its discovery, if we did not know that the simplest ideas are almost always those which are the last suggested to the human mind.

It still remained to combine the principle which has been just explained, with that of virtual velocities, in order to give to the science of mechanics all the perfection of which it appears to be susceptible. This is what Lagrange has atchieved, and by this means has reduced the investigation of the motion of any system of bodies, to the integration of differential equations. The object of mechanics is by this means accomplished, and it is the province of pure analysis to complete the solution of problems. The following is the simplest manner of forming the differential equations of the motion of any system whatever. If we imagine three fixed *(o)* axes perpendicular to each other, and that at the end of any instant the velocity of each material point of a system of bodies is resolved into three others parallel to those axes; we may consider each partial velocity as being uniform during this instant; we can then suppose

that at the end of this instant, the point is actuated parallel to one of these axes by three velocities, namely, by its velocity during this instant, by the small variation which it receives in the following instant, and by this same variation applied in a contrary direction. The two first of these velocities exist in the following instant; the third must therefore be destroyed by the forces which sollicit the point, and by the action of the other points of the system. Consequently, if the instantaneous variations of the partial velocities of each point of the system, be applied to this point in a contrary direction, the system should be in equilibrio, in consequence of all these variations, and of the forces which actuate it. The equations of this equilibrium will be obtained by means of the principle of virtual velocities ; and by combining them with those which arise from the connection of the parts of the system, the differential equations of the motion of each of these points will be obtained.

It is evident that we can in the same manner, reduce the laws of the motion of fluids to those of their equilibrium. In this case, the conditions relative to the connection of the parts of the system are reducible to this, namely, that the volume of any molecule of the fluid remains always the same, if the fluid be incompressible, and that it depends on the pressure exerted according to a *(n)* given law, if the fluid be elastic and compressible. The equations which express these conditions, and the variations of the motion of the fluid, contain the

partial differences of the coordinates of the molecule, taken either relatively to the time, or with respect to the primitive coordinates. The integration of this species of equations presents great difficulties, and we have as yet been only able to succeed in some particular cases, relative to the motions of ponderable fluids in vases, to the theory of sound, and to the oscillations of the sea and of the atmosphere.

The consideration of the differential equations of the motion of a system of bodies, has led to the discovery of several very general and useful principles of mechanics, which are an extension of those already announced in the second chapter of this book, relative to the motion of a point.

A material point moves uniformly in a right line, if it is not subjected to the action of extraneous causes. In a system of bodies which act on each other without being subjected to the action of exterior causes, the common centre of gravity moves uniformly in a right line, and its motion is the same, as if all the bodies were united in this point, all the forces which actuate them being immediately applied (*q*) to it; so that the direction and the quantity of their resultant, remain constantly the same.

We have seen that the radius vector of a body, sollicited by a force, which is directed to a fixed point, describes areas which are proportional to the times. If we suppose a system of bodies acting on each other, in any manner, and sollicited by a force directed to a fixed point; and if from this

point, radii vectores drawn to each of them, be projected on an invariable plane passing through this point, the sum of the products of the mass of each body into the area which the projection of its (r) radius vector traces, is proportional to the time. It is in this that the *principle of the conservation of areas consists.* If there is no fixed point, towards which the system is attracted, and if it be only subjected to the mutual action of its parts, we may then assume any point whatever, for the origin of the radii vectores.

The product of the mass of the body into the area described by the projection of its radius vector in an unit of time, is equal to the projection of the entire force of this body multiplied into the perpendicular let fall from the fixed point, on the direction of the force thus projected; this last product is the moment of the force to make the system revolve about an axis passing through the fixed point, and perpendicular to the plane of projection; the principle of the conservation of areas is therefore reduced to this, namely, that the sum of the moments of the finite forces to make the system revolve about any axis passing through the fixed point, which sum vanishes in the state of equilibrium, is constant in the state of motion. When it is announced in this manner, this principle is applicable in all possible laws between the force and velocity.

The *vis viva* of a system, is the sum of the products of the mass of each body, by the square of its velocity. When the body moves on a curve

or on a surface, without being subject to a foreign action, its *vis viva* is always the same, because its velocity is constant; if the bodies of the system experience no other action, but such as arise from their mutual tractions and pressures, either directly or by the intervention of rods and inextensible and unelastic threads, the *vis viva* of the system remains constant, even though several of the bodies should be constrained to move on curved lines or surfaces. This (*s*) principle, which has been termed the *principle of the conservation of living forces*, is applicable to all possible laws between the force and the velocity, provided that by the *vis viva* or living force of a body, is understood twice the integral of the product of its velocity, into the differential of the finite force by which it is actuated.

In the motion of a body sollicited by any forces whatever, the variation of the *vis viva* is equal to twice the product of the mass of the body, by the sum of the accelerating forces multiplied respectively by the elementary quantities, by which the body advances towards their origins. In the motion of a system of bodies, twice the sum of all these products, is the variation of the living force of the system. Let us conceive that in the motion of the system, all the bodies arrive at the same instant in the position, in which it would be in equilibrio in consequence of the accelerating forces which sollicit it: by the principle of virtual velocities the variation of the living force vanishes; therefore the *vis viva* will then have

attained its *maximum* or *minimum*. If the system be moved by one sole species of simple oscillations, the bodies after departing from the position of equilibrium will tend to revert to it, if the equilibrium be stable; therefore, their velocities diminish in proportion as their distance from this position is increased, and consequently in this position, the *vis viva* will be a *maximum*. But if the equilibrium be not stable, the bodies in proportion as their distance from this position is increased will tend to deviate more from it, and their velocities will continue to increase, consequently their *vis viva* will be in this case a *minimum*. Hence we may infer, that if the *vis viva* be constantly a *maximum*, when the bodies simultaneously attain the position of equilibrium, whatever that velocity may be, the equilibrium will be stable, and on the contrary, the stability will be neither absolute or relative, if the *vis viva* in this position of the system, be constantly a *minimum*.

Finally, we have seen in the second chapter, that the sum of the integrals of the product of each finite force of the system, by the element of its direction, which sum vanishes in the state of equilibrium, becomes a minimum in the state of motion. It is in this (*t*) that the principle of the least action consists, which principle differs from those of the uniform motion of the centre of gravity, of the conservation of areas and of living forces, in this, that these principles are the real integrals of the differential equations of the mo-

tion of bodies; whereas that of the least action is only a remarkable combination of these same equations.

The finite force of a body, being the product of its mass into its velocity, and the velocity multiplied into the space described in an element of time, being equal to the product of this element by the square of the velocity, the principle of the least action may be announced in the following manner: the integral of the *vis viva* of a system, multiplied by the element of the time, is a *minimum;* so that the true economy of nature is that of the *vis viva.* To produce this economy ought to be our object in the con struction of machines, which are more perfect in proportion as less *vis viva* is requir-ed, to produce a given effect. If the bodies are not sollicited by any accelerating forces, the *vis viva* of the system is constant; consequently the system passes from one point to another in the shortest time.

Another important remark remains to be made relative to the extent of these different principles. That of the uniform motion of the centre of gravity, and the principle of the conservation of areas, subsist even when by the (*u*) mutual action of the bodies of the system they undergo sudden changes in their motions, which renders these principles extremely useful in several circumstances; but the principle of the conservation of *the vis viva* and of the least action require, that the variations of the motions of the system be made by insensible gradations.

When the system undergoes sudden changes, either from the mutual action the of bodies of the system, or from meeting with obstacles, the *vis viva* experiences at each of these changes, a diminution equal to the sum of the products of each body into the square of the velocity destroyed, conceiving the velocity previous to the change to be resolved into two, of which one subsists after the shock, the other being annihilated, the square of which is evidently equal to the sum of the squares of the variations which the change makes the decomposed velocity to experience, parallel to any three coordinate axes. All these principles would still obtain, regard being had to the (*v*) relative motion of the bodies of the system, if it was carried along by a general motion common to the foci of the forces, which we have supposed to be fixed. They obtain likewise in the relative motions of bodies on the earth, for it is impossible, as has been already observed, to judge of the absolute motion of a system of bodies, by the sole appearances of its relative motion.

Whatever be the motion of the system and the variation which it experiences from the mutual action of its parts, the sum of the products of each body, by the area which its projection traces about the common centre of gravity, on a plane which passing through this point remains always parallel to itself, is constant. The plane on which this sum is a maximum, preserves its relative position (*x*) during the motion of the system, the same sum vanishes for every plane which passing through the centre of gravity, is perpen-

dicular to that just mentioned; and the squares of the three similar sums relative to any three planes drawn through the centre of gravity, and perpendicular to each other, are equal to the square of the sum which is a *maximum*. The plane which corresponds to this sum, posesses also the following remarkable property, namely, that the sum of the projections of the areas traced by bodies about each other, and multiplied respectively by the product of the masses of the two bodies which are connected by each radius vector, is a *maximum* on this plane, and on all planes which are parallel to it. We may therefore find at all times a plane which passing through any one of the points of the system preserves always a parallel situation; and as by referring the motion of the bodies of the system to it, two of the constant arbitrary quantities of this motion disappear, it is as natural to select this plane for that of the coordinates, as it is to fixtheir origin, at the centre of gravity of the system.

NOTES.

(*a*) The meridian is therefore a secondary both to the equator and to the horizon; and as from Napier's rules the sine of the elevation of any point of the equator above the horizon is equal to the sine of the inclination of the equator to the horizon multiplied into the sine of the arc of the equator, intercepted between the given point and the horizon; it follows, that as the inclination of the equator is constant in the same place, the elevation of the point is greatest when it is 90° from the horizon, *i. e.* when it is on the meridian; in which case also the sine of the greatest elevation of the equator, (which is equal to the complement of latitude,) is equal to the sine of the inclination of the equator to the horizon, and as the most elevated point of the equator exists on the meridian, the most elevated points of all parallels to the equator exist also on the meridian.

As the star is always at the same distance from the pole, when it is on the meridian, it is as much below the pole in one observation as it is above it in the other; hence, the three elevations constitute an arithmetic progression. This observation gives us at the same time the declination, for this last quantity is equal to 90°, minus half the difference between the greatest and least heights;

X

however, this method requires some corrections for refraction, &c. as will be hereafter specified; indeed it has been employed to determine the *quantity of refraction* when the latitude is known from other considerations, (see Brinkley's Astronomy, Chap. 4.); the nearer the star is to the pole the less will be the error from the hypothesis, that there is no refraction; those stars never set, of which the distance from the pole is equal to the complement of the elevation of the pole above the horizon.—*See* Note (*d*), Chap. 2.

(*b*) The actual magnitude of the earth, considered as spherical, may be determined from this circumstance, for if we proceed north or south until the pole is elevated or depressed a degree, we know that we must have travelled over a degree on the earth's surface, the number of miles in which being measured and multiplied by 360, gives the number of miles in the earth's circumference, by means of which it is easy to determine the number of miles in the earth's radius; what is stated in the text shews, that the earth is convex at the place of the spectator; the circumnavigation of the globe in *various* directions proves, that it is a curved surface returning into itself, and likewise the circumstance of the boundary of the earth's shadow in a lunar eclipse being always circular, proves that it is globular or round.

(*c*) The sun's motion is always performed in the same plane; for the sine of right ascension bears to the tangent of declination an invariable ratio, it follows consequently that the plane passingthrough the sun and the vernal equinox must always make the same angle with the equator, the radius being to the tangent of this angle in the given invariable ratio; it is also observed, that the difference between the right ascensions of those stars, which are near to the sun at the commencement of spring and at the commencement of autumn, is 180, hence it follows, that the two intersections of the equator and ecliptic are 180° distant; and if the

points of the horizon, when the sun sets in the beginning of summer and winter, be accurately marked, it will be found that they are equally distant from the east and west, hence, and as all the points of the orbit are always in the same plane, it follows, that the ecliptic is a *great* circle.

(*d*) If l denotes the latitude, d the declination, and h the horary angle from noon, we have, when the sun is rising or setting, $\cos. h = \text{tang}. l. \text{tang}. d$; when the height of the pole and d are of the same denomination $\cos. h$ is negative, and $\therefore h >$ than 90, $\therefore$ the day is longer than the night; when l or d, or both, vanish, $h = 90°$, therefore, the day is always equal to the night; when $\text{tang}. l = \cot. d$, or *vice versâ*, $h = 0$, $\therefore$ the sun does not set.

(*e*) The horizon of spectators situated at the equator passes through the poles, hence the horizon, being in this case a secondary to the equator must pass through the centres of all circles parallel to the equator, and bisect them all at right angles; hence, as also appears from the preceding note, the days are always equal to the night; such a position of the sphere is called a right sphere. To a spectator situated at the pole, the equator and horizon coincide, consequently the planes of all the diurnal circles are parallel to the plane of the horizon, so that when the sun is at the northern side of the equator, he does not set for six months; this position of the sphere is called a parallel sphere. In all places intermediate between the equator and poles, the length of the day is different at different periods of the year. Such positions of the sphere are called oblique spheres; what is stated here is immediately apparent from the preceding part of this note (*d*).

If by means of the observed declinations and right ascensions of the sun, the daily increments of longitude be computed, it will be found that they are not proportional to the intervals of time which separate the consecutive passages of the sun over the meridian; the greatest difference exists in two points of the ecliptic, of which one is situated

near to the summer, and the other near to the winter solstice; these two points are in the same line, though situated on opposite sides of the equator, and their right ascensions differ by 180.

(*f*) Consequently the mean velocity between these two extremes is $\frac{1^\circ,1327+1^\circ,0591}{2}=1^\circ,0959$.

(*g*) The angles being supposed to increase proportionably to the times, their sines will be periodical; for the sine, which at the commencement is cypher, increases with the arc and becomes equal to radius when the arc $= 90^\circ$, it then decreases and finally becomes = to cypher when the arc becomes equal to 180; the sine then passing to the other side of the diameter changes its sign, and runs through the same series of changes in this semicircumference. It may be remarked here, that it appears from analysis that all the inequalities of the heavenly bodies may be expressed by the sines and cosines of angles, which increase proportionally to the time. No other function of the circle occurs in the expressions for these inequalities. *See* Vol. 2, Book 6, Chap. 2.

(*h*) In fact, as it is a matter of observation that the angular motion of the sun varies as the square of the apparent diameter, it follows, as a general law, that the angle described each day by the sun multiplied by the square of the distance is constant, *i. e.* if r and r' represent the distances, and dv, dv' the angles described by the sun at the two different epochs, we have $dv.r^2=dv'.r'^2$; but the areas described at these points are respectively $=\frac{dv.r^2}{2}$, $\frac{dv'.r'^2}{2}$, hence it follows, that equal areas are described in equal times.

Otherwise thus, let v and v' represent the angular motions of the sun in two different points of the orbit, a and a' the small diurnal arcs described by the sun at these points, r and

$r+\delta r$, the corresponding distances, and d, d' the corresponding apparent diameters, the small sectors described by the sun are equal respectively to $\frac{r.a}{2}$ and $\frac{(r+\delta r).a'}{2}$, or as $a=vr$, $a'=v'.(r+\delta r)$, these sectors are equal to $\frac{r^2.v}{2}$, $\frac{(r+\delta r)^2.v'}{2}$; now by means of very exact measurements of the apparent diameter of the sun made with a micrometer, it is found that the apparent angular motions vary as the squares of the apparent diameters, i. e. $v:v'::d^2:d'^2$, or $v:v'::(r+\delta r)^2:r^2$, $\therefore vr^2=v'.(r+\delta r)^2$, hence the small sectors are always proportional to the times.

(i) In fact, suppose lines to be drawn in a plane passing through a given point, (which represents the common centre of the earth and of the celestial sphere,) so that their angular distances may be equal to the diurnal motion. These lines will represent the visual rays, which are drawn to the sun each successive day. Lay off from the fixed point in the direction of these rays the corresponding distances of the sun from the earth, (which may be estimated from the diurnal motion, one of these distances being assumed equal to unity,) the points which are determined in this manner will indicate the place of the sun for each day, and the curve which is traced by uniting these points will be similar to the sun's orbit. It is evident, that if the angles described by the sun each succsssive day be determined by means of its observed longitudes, the ratio of the distances will be obtained; for, from the equation $vr^2=A$, it follows, that these distances are reciprocally as the square roots of the angular motions. But in order to ascertain whether the curve indicated by the observations of the sun is an *exact* ellipse, we should assume the indeterminate equation of any ellipse, and make it to satisfy some of these observations; and when the elements have been determined by this condition, we can investigate and try whether it equally represents the other observations,

i. e. if it assigns for the distances of the sun from the earth in different longitudes, values equal to those which have been deduced from observation.

We might have inferred from an observation of the sun's apparent diameter that his apparent orbit is an ellipse, for if m be his mean, and $m—n$ his least apparent diameter, then this diameter at any other point is observed to be equal to $m—n.\cos.v$, v representing his angular distance from the point where his diameter is least; now, as the distance varies inversely as the apparent diameter, $r=\frac{\mathrm{B}}{m—n.\cos.e}$, which is an equation of the same form as $r=\frac{a.(1—e^2)}{1+e.\cos.v}$.

Or thus, let D, D′ represent the greatest and least diameters of the sun, which have been already given in numbers in the text; it is found that if d denote any other diameter, and v the angular distance of the sun when the diameter is d, from the point in the ellipse where the diameter is D, we have $\mathrm{D}-\mathrm{D}' : \mathrm{D}-d :: 1—\cos.180\ (i.\ e.\ 2) : 1-\cos.v$, $\therefore (\mathrm{D}-\mathrm{D}')(1-\cos.v) = 2.(\mathrm{D}—d)$ and $d=\mathrm{D}—\frac{(\mathrm{D}—\mathrm{D}')}{2}.(1-\cos.v) = \frac{\mathrm{D}+\mathrm{D}'}{2} + \frac{\mathrm{D}—\mathrm{D}'}{2}.\cos.v$, $\therefore \frac{1}{r} = \frac{1}{a.(1-e^2)} + \frac{e}{a.(1-e^2)}.\cos.v = \frac{1+e.\cos.v}{a(1—e^2)}$, this is the equation of an ellipse whose major axis passes through the points where the apparent diameter is greatest and least.

(*k*) This point may be easily determined in the case of any elliptic orbit. About the focus of the ellipse, as centre, describe a circle, of which the radius is a mean proportional between the semiaxes of the ellipse; this circle is equal to the ellipse, and if a body be conceived to revolve in this circle with the mean angular motion of the

sun, its periodic time will be equal to the periodic time of the sun. Conceive this imaginary body to set off from the same radius as the sun, at the same time that the sun begins to move from the apogee. As the sun's velocity in this point is less than his mean angular velocity, the fictitious body will precede the sun, and it will continue to precede this star by greater quantities perpetually, till the angular motion of the sun becomes equal to the angular motion of this body, afterwards the angular motion of the sun becoming greater than the mean angular motion, the sun will begin to gain on the body, and will overtake it, when it arrives at perigee; hence it is evident, that the body precedes the sun by the greatest quantity, when its angular motion is equal to the mean angular motion; now it appears from the equation $vr^2 = A$, that the angular motions vary as the synchronous areas directly, and inversely as the squares of the distance, but the synchronous areas are equal in the ellipse and circle, for they are as the whole areas divided by the respective periodic times, *i. e.* in a ratio of equality, hence, the angular motions are equal when the distances are equal, *i. e.* when the distance of the sun from the focus is a mean proportional between the semiaxes.

The radius of the circle whose area is equal to that of the ellipse $= a . \sqrt{a^2 - e^2}$.

(*l*) This parallax is given with great accuracy by theory, as we shall see in the sequel, (see Book 4, Chap. 4,) the reason why it is so particularly interesting to determine the parallax is, because our knowledge of the absolute magnitude of the solar system depends on it.

If the exact time when the spots describe right lines was known, the longitude of the sun or earth at this instant would determine the place of the nodes. However, this place is best determined by means of corresponding observations, made before and after the passage through

the nodes when the openings of the ellipse is the same, but in opposite directions.

Calling l, x the heliocentric longitudes of the earth and spot, y the heliocentric latitude, and B the geocentric latitude, Δ the sun's semi-diameter, r the distance of spot from centre of the sun, and R the distance of spot from centre of the earth, which is very nearly equal to the distance of the centre of the sun from earth, we have $r : R :: \sin. B : \sin. y$, $\therefore$ as $\sin. \Delta = \frac{r}{R}$, we have $\sin. y = \frac{R}{r} \cdot \sin. B = \frac{\sin. B}{\sin. \Delta}$, likewise $r. \cos. y : R. \cos. B$ expresses the ratio of the curtate distances of the spot from the centres of the sun and earth, which is also expressed by that of $\sin. E : \sin. (l - x)$, E being equal to the difference between the geocentric longitudes of the centre of the sun and spot, $\therefore$ we have $r. \cos. y : R. \cos. B :: \sin. E : \sin. (l - x)$, hence $\sin. (l - x) = \frac{\sin. E. \cos. B}{\cos. y} \cdot \frac{R}{r} = \frac{\sin. E. \cos. B}{\cos. y. \sin. \Delta} =$ (by substituting for $\cos. y$ its value $\frac{\sqrt{\sin.^2 \Delta - \sin.^2 B}}{\sin. \Delta}$) $\frac{\sin. E. \cos. B}{\sqrt{\sin.^2 \Delta - \sin.^2 B}}$, hence we can determine x.

Observing three positions of the same spot, we are given by what precedes their distances, l, l', l'', from the pole of the ecliptic or their co-latitudes. We can also, by what precedes, determine their differences of longitude; hence in the three spherical triangles, which are formed by drawing lines from the pole of the ecliptic to the three observed positions of the spot, we have in each of them, respectively, two sides and the included angle, which enables us to determine the remaining sides, A, A′, A″, (or the arcs connecting the three positions of the spot), and also the base angles, and consequently their sum; now as the spot moves parallel to the solar equator, its distances from the pole of this equator are the same, consequently a perpendicular from this pole bisects the arcs connecting the different

positions of the spots; and from the consideration of these triangles it is evident, that we are given the ∺ of the sum of the cosines of the angles, which an arc from the pole of the sun's equator makes with the connecting arcs, to the difference of the cosines of these angles, *i. e.* we are given the ratio of the cotangents of half the angle made by connecting arcs to the tangent of half the difference of the preceding angles; having determined this difference we can obtain the angle which the arc from the pole of the equator makes with connecting arc, and hence we obtain, by subtraction, the angle formed by arcs drawn from a given position of the spot to the poles of equator and ecliptic, and as we have these arcs we can obtain the third side, which measures the inclination of the equator to ecliptic; and as we also know a, the angle formed by l, l', and the time, t, in which it is described, we can obtain the time of revolution for $t : T :: a : 360°$.

CHAPTER III.

This position, with respect to the equality which subsists between the duration of each oscillation of a pendulum, is, in fact, the principle of sufficient reason which was first propounded as a general axiom by Leibnitz, though it was long before *virtually* assumed by Archimedes in demonstrating some of the first principles of mechanics.

The sun in the course of the year passes the meridian once less than the star, because the sum of all the retardations in that time is equal to 360°, being equal to the sum of the arcs described by the sun in the year, *i. e.* to 360.

It may be remarked here, that in consequence of the

precession of the equinoxes, the star takes a longer time to return to the meridian than the revolution of the earth on its axis; however, the difference is not appreciable, for supposing that the annual precession in right ascension is $50''1$, which it is very nearly for stars near the equator, this converted into time gives 3,3 seconds, by which the star passes the meridian later at the end of a year, which being distributed over the entire year is altogether insensible.

(m) Let I be the obliquity of the ecliptic, l the longitude of the sun, and A the right ascension; then if $\cos. \mathrm{I}=s$, $\mathrm{tang}. l=x$, we have, by Napier's rules, $sx=\mathrm{tang}. \mathrm{A}$, $\therefore$ $s.dx$, i. e. $dl.(1+x^2).s = d\mathrm{A}(1+\mathrm{tang}.^2\mathrm{A})$, or $sdx = s.dl.\sec.^2 l=d\mathrm{A}.\sec.^2\mathrm{A}$, that is, $\dfrac{dl\ s}{\cos.^2 l}=\dfrac{d\mathrm{A}}{\cos.^2\mathrm{A}}$, and since $\cos. l=\cos.\mathrm{A}.\cos.\mathrm{D}$, (D being equal to the declination,) we obtain $\dfrac{dl.s}{\cos.^2\mathrm{A}.\cos.^2\mathrm{D}}=\dfrac{d\mathrm{A}}{\cos.^2\mathrm{A}}$; therefore, $d\mathrm{A}$ (which converted into time determines the variation of the astronomical day,) is equal to $d.l.\sec.^2\mathrm{D}$; and as dl and s are constant, $d\mathrm{A}$ varies as $\sec.^2\mathrm{D}$, and therefore it is greatest at the solstice, and least at the equinox; for $d\mathrm{A}=dl.s$ at the equinox, and at the solstice $d\mathrm{A}=\dfrac{dl}{s}$, $\therefore$ $d\mathrm{A}$ at the equinox is to $d\mathrm{A}$ at the solstice as $s^2 : 1$, consequently dl is a mean proportional between the increment in the equinoxes and in the solstices; l is evidently equal to the right ascension of the fictitious sun s'', which is supposed to move in the equator with a motion equal to the sun's mean motion in the ecliptic; $\therefore$ $l-\mathrm{A}$ is equal to the separation of s'' from s', and $\tan.(l-\mathrm{A})= \dfrac{\tan. l-\tan.\mathrm{A}}{1+\tan. l.\tan.\mathrm{A}}=\dfrac{x-sx}{1+sx^2}=(1-s).\dfrac{x}{1+sx^2}$, $\therefore d.(l-\mathrm{A}) = (1-s)\,dx.\dfrac{(1-sx^2)}{(1+sx^2)^2}$; which is a *maximum* when $1=sx^2$, that is, $\tan. l=\dfrac{1}{\sqrt{\cos.\mathrm{I}}}$, and consequently $\sqrt{\cos.\mathrm{I}}=\tan.\mathrm{A}$,

hence $A = 43°, 43', 56''$, and $l = 46°, 14'$, and $l - A$ when a *maximum* $= 2°, 28', 20''$; it appears from this that the greatest separation of s'' from s' is greater than the greatest separation of s' from s on the equator, corresponding to the greatest equation of the centre, for the latter is only $2°, 6'$, when the greatest equation is $1°, 55' 33''$; besides, this greatest separation happens about the 8th of May, which is later than when the radius vector of the solar orbit is a mean proportional between the semiaxes, that is, when the equation of the centre is *maximum.*—*See* Note (k), page 302.

(n) Hence, as the second and third sun's depart from the equinox together, the one describing the equator, and the other the ecliptic, with the same uniform motion; the distance of the latter (which is equal to the mean longitude of the true sun) will be equal to the right ascension of the third sun. Hence the equation of time may be defined to be the *difference between the true sun's right ascension and his mean longitude, corrected by the equation of the equinoxes in right ascension*; therefore, naming e the equation of the centre, μ, v the increments in longitude and right ascension which result from the nutation, r the reduction to the equator, or the difference between the longitude and right ascension, λ', λ the true and mean longitudes of the sun, ρ' ρ the true and mean right ascensions, and p the effect produced by the perturbations of the planets, we have $\lambda' = \lambda + e + p + \mu$, $\rho = \lambda + v$, $\rho' = \lambda' + \rho = \lambda + e + p + \mu + \rho$, $\therefore \rho' - \rho = e + p + r + \mu - v$; we will see hereafter that $v = \mu . \cos . \varepsilon$ (ε being the obliquity) and $\mu = 18'' . \sin . \zeta$, ($\zeta$ depending on the situation of the lunar orbit), therefore x, the equation of time, $= \frac{e+p+r}{15} + \frac{18 . \sin . \zeta}{15} \cdot (1 - \cos . e)$; since, therefore, both e and r are variable in this expression, the equation must (without taking into account the disturbing force μ or p) be variable from these two causes; and as e and r are not the same on corresponding days of

two different years, in consequence of the secular disturbances, it follows, that the equation of time is continually varying.

There are four times in the year in which the equation of time vanishes, for denoting the true sun, the sun which moves with a mean motion in the ecliptic, and the sun which moves with a mean motion in the equator by s, s', s'' respectively. As s' precedes s from apogee to perigee, and s'' precedes s' from the autumnal equinox to the solstice, the order of the sun's near the winter solstice is s, s', s''; at the solstice s' coinciding with s'' the order is $s, \left\{\frac{s'}{s''}\right\}$ immediately after s' passes s'', (as appears from what has been established above respecting the increments of $d\Lambda$, at the equinox and at the solstice), ∴ after the solstice the order is s, s'', s'; at the perigee, which is very little beyond the solstice, s coincides with s'; ∴ it must have passed s'' in order to effect this, for s'' does not overtake s' till their arrival at the vernal equinox; hence, at the moment when s passed s'', the equation of time vanished. After the perigee the order of the sun's is $s''s's$, which continues to the vernal equinox, therefore in that interval the equation of time does *not* vanish; after the equinox s'' begins to precede s', and the order becomes $s's''s$; very near this point the distance from the focus of the solar ellipse is a mean proportional between the semiaxes, *i. e.* the true angular motion is equal to the mean angular motion, and therefore s' is at the greatest distance from s. But the greatest separation of s'' from s' is subsequent to this, and as it is greater in quantity than the deviation of s from s', it follows, that previous to the greatest separation of s'' from s', the order of the sun's is not $s's''s$ but $s'ss''$; therefore s'' must have passed s, consequently the equation of time must have vanished; but at the summer solstice s'' joins s', and as s' does not join s till after the time of the solstice, when the sun is in the apogee, this

junction of s'' with s' must have been effected by s'' repassing s, this caused the equation of time again to vanish, previous to the time of the solstice; after this takes place the order of the sun is $s's''s$, at the solstices s' coincides with s'', and after this the order becomes $s''s's$ till the sun arrives at apogee. Immediately after s' moving with a greater angular motion than s, the order becomes $s''ss'$; now as s cannot overtake s' till it arrives at perigee, whereas s'' reaches s' at the equinox, it follows, that previous to this s'' must have passed s, and at the instant of passing, the equation of time vanishes. If the apogee and perigee coincided with the solstices, the equation of time would vanish in these points, which was the case in the year 1250; but as the apsides continually prograde, the points, at which the equation of time vanishes, continually vary. As the moments when the equation from each cause separately is a maximum, do not coincide, the greatest equation can never be equal to the sum of the two equations arising from each cause separately; when the equation of time is a *maximum*, its increment is cypher, *i. e.* the mean and true day have the same length, when the equation of time vanishes, their difference is the greatest possible.

(*o*) The reason why the day was divided into 24 hours, and the hours into 60 minutes, and the minutes into 60 seconds, was, because these numbers admitted many different divisors.

If the year was exactly $=365+\frac{1}{4}$, in four years the commencement of the year would have regraded an entire day, and in 1460 *Julian* years the commencement would have regraded an entire Julian year, for dividing 1460 by by 4, the quote will be 365, $\therefore$ 1460 Julian are equivalent to 1461 Egyptian years, but as the year is accurately only $=365,2422640$, in order that the difference between this and 365 may produce a tropical year, it is necessary that 1508 years should be accomplished; this period of 1460

is called the *sothiac* period. The Egyptians supposed all their months to consist of 30 days, and they added at the end of the year five days, which were called ἐπαγομενα. *See* Vol. 2, Book 4, Chap. 3.

(*p*) Suppose that the moment of the solstice or equinox preceded midnight by a quantity less than the errors of the tables, then according to the tables the moment would happen *after* midnight, and as the commencement of the year is reckoned from *the midnight* which precedes the solstice *as* determined by the *tables*, this origin would differ nearly by an entire day from the true origin.

(*p*) In the *Julian* arrangement of the year, it is supposed that $365+\frac{1}{4}=R$, a revolution of the sun; consequently, though there is not an integral number of days in *one* revolution, still four years may be made equal to four revolutions of the sun, and $4R=4.365+1=3.365+366$; now as the true length of the year is not 365.25, but 365.242264, which is less than the former by $11', 15''$; before a new year has commenced, the sun has passed the point in the ecliptic where the last year began, by a small fraction $= 11', 15'' \times 59', 8''$; therefore, the Julian reckoning and the course of the seasons fall behind the sun, and in 132 years this difference is very nearly a day, hence in 3.132 or 396, which is nearly equal to four centuries, their loss would be three days; this is the reason why Gregory proposed to omit the intercallary day at the commencement of three successive centuries, which would be, in the Julian arrangement, intercallary years, and to retain it on the fourth century, and two hours fifteen minutes is all that remains uncorrected; for $11', 15''$, the annual error $= .007736$, which in a century is $.7736$, and in four centuries it is equal to 3.0944, of which the decimal part $.0944$, which is not corrected, $= 2^h, 15'$; now this part becomes equal to an entire day in 4237 years, and therefore it would be corrected nearly by omitting, as

is suggested in the text, a bissextile every four thousand years.

(*q*) The Persian intercalation was more correct than the Julian, for Omar proposed to delay to the 33d year the intercalation which ought regularly to take place on the 32d, and by this means the Julian intercalation would be altogether omitted in the 128th year; but it has been before observed, that the Julian intercalation is too much by one day in 132 years, this method is therefore more exact than that proposed by Gregory, for it differs from the truth only by one minute in 120 years; (in fact, if we determine the series of continued fractions which express the ratio between $5^h\ 48'\ 49''$ and 24^h, the first terms of the series are $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{31}{128}$, and among the terms of this series the ratio, which would exist according to the intercalation proposed by Gregory, does not occur; $\frac{1}{4}$ is greater than the true difference, and $\frac{7}{29}$ less, therefore, as the fractions converge towards the true value, the correction proposed by Omar is more accurate than Cæsar's or Gregory's).—*See* Vol. 2, Book 6, page 220.

(*r*) The order of the planets, according to the ancients, is Saturn, Jupiter, Mars, the Sun, Venus, Mercury, Moon; now the names of the planets are imposed on the days *δια τεσσαρων*, *i. e.* as the sun is the fourth from Saturn inclusively, he denominates the first day of the week; the moon being the fourth from the sun, denominates the second day of the week, and so on. An astrological reason has also been assigned; for as the planets were supposed to preside over *each* hour of the day, and as the planet gave its name to that day, over the first hour of which it presided, if the sun would have the first hour it would have also the 8th, the 15th, and in general all those of the form $1+7n$; Venus would preside over the second hour, and in general all those of the form $7n+2$; Mercury over all those of the form $7n+3$; the moon over those of the form $7n+4$; Saturn over those of the form $7n+5$; Jupiter over those of

the form $7n+6$; Mars over those of the form $7n+7=7.(n+1)$. The general formula is $7n+a=24m+1$, therefore $a-1=24m-7n$ if $7n+a=1$; $n=0$; $a=1$; the first day belongs to the sun; $7n+a=25$, n must be equal to 3, therefore $24m-7n=4$, therefore the moon presides over the second day; if $7n+a=73$; $a=73-7.10$, therefore $a=3$, and the fourth day will be that of Mercury; and after the seven planets are exhausted, the days will return in the same order as before; for let $7n+a=169$, therefore $a=169-7.24=1$, therefore the eighth day belongs also to the sun as well as the first.

CHAPTER IV.

Calling M the diurnal motion of the moon, μ the diurnal motion of the sun, $(M-\mu)$ will be the relative motion with which the moon regains the sun, ∴ we shall have $M-\mu : 1^\circ :: 360 : L=\frac{360^\circ}{M-\mu}$, as we know very nearly the duration of a synodic revolution, we know the number of synodic revolutions in a given interval N; hence, if n represent this number, we have $n.\left(\frac{360}{M-\mu}\right) = N$, therefore $\left(\frac{n}{N}\right)\cdot 360^\circ=M-\mu$, which is the relative motion, hence we can determine M and $\left(\frac{360}{M-\mu}\right)$; now if m represent the mean motion of the sun, $n.360+m:360::N:\frac{N.360}{n.360+m}=$

$$\frac{\left(\frac{N}{n}\right)}{1+\frac{m}{n.360}}=\left(\frac{N}{n}\right)\left(1-\frac{m}{n.360^\circ}+\left(\frac{m}{n.360^\circ}\right)^2-\&c.\right)=P=$$

the revolution in longitude of the moon, in order to determine P′, the sidereal revolution, we have $360°-p$: $360° :: P : P' = \frac{P}{1-\frac{p}{360°}} = P.\left(1+\frac{p}{360}+\left(\frac{p}{360}\right)^2+\&c.\right)$, ($p$ being the precession in a tropical month), if M represent the motion of the apsis during P, we have $360°-M ::$ $360 :: P : P''$ the anomalistic revolution, $= \frac{P}{1-\frac{M}{360}} =$ $P\left(1+\frac{M}{360}+\left(\frac{M}{360}\right)^2-\&c.\right)$ Let M′ represent the motion of the node, and as it regrades we have $360+M' : 360 ::$ $P : P''' = P.\left(1-\frac{M'}{360}+\left(\frac{M}{360}\right)^2-\&c.\right)$ hence it appears how all these different periods may be readily inferred from the synodic revolution, which may be accurately determined by means of two eclipses separated by a considerable interval from each other.

The orbit of the moon may be proved to be elliptical in the same manner as the sun's orbit was shewn to be elliptical.

(s) Let D, D′ represent the greatest and least apparent diameters of the moon, the eccentricity $=\frac{D-D'}{D+D'}$, and if p be the horizontal parallax when the moon's apparent diameter is d, the parallax at the least distance $=\frac{p.D}{d}$, and at the greatest distance $=\frac{p.D'}{d}$, therefore the least distance $=\frac{r.d}{p.D}$, and the greatest distance $=\frac{r.d}{p.D'}$, consequently the mean distance on the hypothesis that the orbit is elliptical $=\frac{r}{2}.\frac{d}{p}.\left(\frac{1}{D}+\frac{1}{D'}\right)$; r represents the radius of the earth, supposed spherical; but on the supposition that the earth is an ellipsoid, r is the radius corres-

ponding to the latitude, of which the square of the sine $=\frac{1}{3}$. *See* Chap. 4.

From the eccentricity, the equation of the centre may be inferred by means of the formula

$$(2e-\tfrac{1}{4}e^3+\tfrac{5}{96}.e^5).\sin.nt+(\tfrac{5}{4}e^2-\tfrac{11}{24}e^4+\tfrac{11}{192}.e^6).\sin.2nt+ (\tfrac{13}{12}e^3-\tfrac{43}{64}.e^5).\sin.3nt+\&c.$$

See Notes to Book 2, Chap. 3. Hence it is easy to shew when it will be a maximum.

(t) If ☾ be the mean longitude of the moon, and ☉ that of the sun, the mean anomaly being nt, as before, this in equality is

$$(1^\circ.21,5''.5)\sin.2(☾-☉)-nt));$$

hence, as ☾ —☉=0, or 180°, in the oppositions or conjunctions of the moon with the sun, the argument in these positions is $-nt$, which renders the evection negative, if nt is <180, and positive if nt is >180, contrary to what happens in the equation of the centre, as is evident from an inspection of its value; therefore, in both cases it is diminished. The period of the evection may be inferred from the rate of increase of its argument, which is $11^d.3166$ per day, its period therefore is $\frac{360}{11.3166}$, or $31^\circ.8119$. The evection may be considered as an inequality in the equation of the centre, arising from an increase of the eccentricity at the quadratures, and a diminution of it at the syzygies; it appears from its argument that it depends on the position of the axis major of the moon's orbit, with respect to the line connecting the sun and earth.—*See* Princip. Math. Lib. I., Prop. 66, Cor. 9.

(t) This inequality may therefore be represented by the formula $(35'\,42'').\sin.2(☾-☉)$. Its period is evidently equal to $14^d.7655$, or half a lunar month. Mayer has added to the preceding value of the variation two other

terms, which are respectively proportional to sin. $3(☾-☉)$, sin. $4(☾-☉)$.—*See* Notes to Vol. II. Chap. 4.

(*u*) The argument of this inequality is sin. mean. Anom. ☉. Hence, in the eclipses it is confounded with the equation of the centre of the sun. This equation arises from the variation of the sun's distance from the earth.—*See* Notes to Vol. II. Chap. 4.

(*v*) During *each* revolution of the moon the nodes advance, and regrade alternately; but the quantity of the regress exceeding that of the advance, the nodes during a revolution may be said on the whole to regrade, as the excess of the arc of regression above the arc, during the description of which the nodes advance, is twice the distance of the node from syzygy, the regress of the nodes will increase in the passage of the nodes from syzygy to quadrature, and again decrease in the passage from quadrature to syzygy.

Let l, l' represent two latitudes of the moon on successive days, before and after passing the node, λ, λ' the corresponding longitudes, and n the longitude of the node, and I the inclination, we have $l+l' : l : \lambda-\lambda' : n-\lambda = \frac{l.(\lambda-\lambda')}{l+l'}$, hence we get n, and as by Napier's rules $\sin.(\lambda-n)=\tan.l.\cot.I$, we obtain $\cot.I=\sin.(\lambda-n).\cot.l$, I might also be obtained by observing the moon's latitude on several days near to its maximum, for the greatest latitude is evidently $=$I.

Since the argnment of the greatest inequality is proportional to the sine of double the distance of the sun from the ascending node of the lunar orbit, its period must be equal to a semi-revolution of the sun with respect to the nodes of the moon.

(*w*) It was from the variation of the moon's apparent diameter that Newton inferred that the areas were proportional to the times.—*See* Princip. Lib. 3, Prop. 3.

(*x*) The lunar inequalities have been distinguished into

three classes, namely, those which affect the longitude, those which affect the latitude, and those which affect the radius vector of the moon. The reason why it was so easy to discover them, was because their periods were of such different durations. With respect to the inequalities which affect the longitude of the moon, three, namely, the evection, variation, and annual equation, have been known to the ancient astronomers; but there are several others, the existence and form of which have been indicated by theory, and which may be considered as so many corrections to be applied to the above mentioned inequalities, in order to determine the position of the moon with the accuracy required by the precision of modern observations. It is the same with with respect to the inequalities which affect the latitude and the radius vector of the moon. The forms of the inequalities are determined by physical astronomy; the coefficients are determined by observing when they attain their greatest values, for then the angular functions into which they are multiplied are equal to unity.

(y) The greatest breadth of the illuminated part of the moon's surface is observed to vary as the versed sine of the moon's elongation; but if the moon was spherical, the illuminated part would vary as the versed sine of the exterior angle at the moon, which differs very little from the angle of elongation. Strictly speaking, the illuminated portion varies as the versed sine of the exterior angle at the moon $=$E, (the angle of elongation) $+$(S) angle at the sun; this last quantity, or its sine, which is nearly the same thing, $=\sin. E \times$ into the $\div$ of the moon's distance into the sun's distance from the earth, $=\frac{60.r}{d}\cdot\sin. E$, where r represents the rad. of the earth, d the distance of sun from earth, but $\frac{r}{d}=8''.47$ the sun's parallax, therefore $S=8'.47.\sin. E$ and $P=\Delta$ versed sine $(E+8'.47.\sin. E)$; now

from Napier's rules cos. E$=$cos. l. cos. ($☽ - ☉$); l and ☽, ☉, representing the same as in the preceding notes, in conjunction $☽ - ☉ = 0$ and $E = l$; therefore $P = \Delta$ vers. sine $(l + 8'.47.\ \sin. l)$, $\therefore$ unless the moon is in its node, the illuminated part does not vanish; when $E + 8'.47.\ \sin. E = 90°$, $P = \Delta \frac{1}{2}$, $\therefore$ half the disk is illuminated; in this case E is less than 90°, as stated in the text, when $☽ - ☉ = 90$, $E = 90°$, and P is greater than Δ; when $☽ - ☉ = 180$, $P = \Delta$. versed sin. $(180° - l - 8'47.\ \sin. l)$; and when l vanishes, *i. e.* at the node, $P = 2\Delta$; calling $2\Delta'$ the apparent disk of the earth as seen from the moon, $180° - E$ is the exterior angle at the earth, and P' the illuminated part $= \Delta'$. ver. sin. $(180 - E) = \Delta'.(1 + \cos. E)$, but $P = \Delta$. ver. sin. $(S + E) = \Delta.(1 - \cos. E)$, nearly; hence, when $E = 0$, $P = 0$, and $P' = 2\Delta'$, and when $E = 180°$, $P = 2\Delta$, $P' = 0$.

If the angular motion of the moon was exactly equal to that of the sun, the lines drawn from the earth to the sun and moon would preserve the same relative position, and the moon would invariably present the same aspect, the quantity of the illuminated surface being always the same.

(z) It appears, therefore, that the period of the phases is the time required to describe four right angles with an angular motion, equal to the difference between the angular motion of the sun and of the moon, it is consequently greater than the time of tropical revolution.

(a) This is the method employed by Aristarchus to determine the distance of the sun from the earth, and is the first attempt on record to determine this distance.

(c) Half the angle of this cone is equal to semid. ☉—parallax ☉; therefore, if r be the radius of the earth, s the apparent semidiameter, and p the horizontal parallax of the sun, the height of this shadow reckoned from the earth's centre $= \frac{r}{\sin.(s - P)}$, and the semiangle of the sec-

tion of the shadow $=P+p-s$; P representing the horizontal parallax of the moon.

(*d*) The ecliptic limits, or the greatest distance from the node at which an eclipse can happen, is determined by computing the moon's distance from the node, when she just touches the earth's shadow; we might by a similar manner compute the limits of a *total* eclipse.

When the angle at the moon is 90, the moon must be dichotomized, and therefore the boundary of the illuminated part is a right line; and conversely when the boundary is a right line, the angle at the moon is a right angle, therefore in this case the sun's distance from earth is to moon's distance from earth, *i. e.* moon's parallax : sun's parallax as 1 : cos. elongation.

The rad. of the penumbra $P+p+s$, therefore we might compute the time of the moon's entering and emerging from the penumbra. As the earth's atmosphere intercepts some of the rays of light coming from the sun, it causes the shadow of the earth to appear somewhat greater than it would be if there was no atmosphere, the parallax of the moon ought, according to Mayer, to be increased its sixtieth part.

The ecliptic limits for the sun may be computed in a manner similar to that for computing the ecliptic limits of the moon, and as they are greater than those of the moon, there are more solar eclipses than lunar in a year, though more lunar eclipses are visible at any given place.

(*e*) The ray of light at its entrance into the lunar atmosphere is inflected towards the perpendicular, and it suffers an equal deflection from the perpendicular at its egress; each of these deviations is equal to the horizontal refraction of the lunar atmosphere, so that the entire inflection of the ray equals very nearly twice the horizontal refraction. Hence the star continues visible some time after the moon has been actually interposed between the

star and observer; and it is also, for the same reason, seen some time before it ought to be visible, from which it follows, that the duration of an occultation of a fixed star by the moon is less than if there was no lunar atmosphere; however, as the entire duration is never lessened eight seconds of time, the beginning of the occultation will not be retarded, nor the end of it accelerated by four seconds of time; if the retardation was four seconds of time, the horizontal refraction would be two seconds of space, for the moon moves over 2″ of space in 4″ of time; therefore as the densities are proportional to the horizontal refractions, the density of the lunar atmosphere is 1000 times less than the density of the terrestrial atmosphere, which is a density much less than what can produced in the best constructed air pumps. And as without the pressure of the terrestrial atmosphere, all the liquids which at present exist on its surface would be dissipated into vapours, (see Chap. 16, Book 1,): the pressure of the lunar atmosphere being so very inconsiderable, it follows, that if there was any large collection of water on its surface, it would long since have been dissipated. Besides, if there was a quantity of water spread over the lunar surface, whenever the circle of light and darkness passed through it, it would exhibit a regular curve.

(*f*) Bouguer found that if the light of the sun, when elevated 31° above the horizon, and introduced into a darkened chamber, be made to pass through a concave mirror, it would be dilated into a space of 108 lines of diameter, or weakened 11664 times, and in this state it was equivalent to the light of a candle 16 inches distant. The light of the moon when full, and at the same elevation above the horizon, was found to be dilated into a space of eight lines of diameter, or weakened 64 times, which is equivalent to the light of the same candle when it is distant fifty feet. Thus the light of the sun when enfeebled 11664 times, was still 443 times stronger than the light of the moon

when rendered weaker only 64 times. Hence the ratio of the one one to the other is about that of 1 to 268000. Other observations made the ratio that of 1 to 300,000, which is very nearly the mean of several observations. A different estimation is given in Smith's Optics.—*See* Young's Analysis, p. 305.

(*g*) The part of the moon in which this light is visible corresponds exactly to the part of the moon which is not illuminated by the sun; which is exactly equal to the part of the earth which would appear to a spectator on the moon illuminated by the sun.

(*h*) If the axis of rotation of the moon was in the plane of the moon's orbit, every part of the moon would be successively presented to the earth, though the moon revolved on her axis in the time of her revolution about the earth; so that the perpendicularity of the axis of rotation to the plane of the orbit is a condition, which must be combined with the equality of the times of rotation and revolution, in order that the same face may be always presented to us. If the axis of rotation was exactly perpendicular to the plane of the moon's orbit, the libration in longitude would be a *maximum* at the point where the equation of the centre was greatest, (see page 302). From apogee to this point parts of the western edge of the moon come into view, and from this point to perigee these parts are gradually restored; the contrary takes place in the other half of the orbit. The libration of a spot towards the centre of the lunar disk, is much more sensible than the libration of a spot near to the border.

It appears from what is stated in the text, that there are four kinds of librations of the moon; three apparent, one real.

(*i*) The axis of rotation remains parallel to itself, making with the plane of the ecliptic an angle of 88½°, and therefore with the plane of its orbit, which is inclined to that of the ecliptic in an angle of 5°, 10′, an angle of 83°

at its greatest latitude. The descending node of the lunar orbit coincides with the ascending node of the moon's equator. The axis of the earth being inclined to the plane of the ecliptic at an angle of 66°, 23′, the earth must exhibit to a spectator at the sun, appearances similar to those which the moon presents to us, *i. e.* at the time of the summer solstice a portion of its disk about the north pole of 23°, 28′ extent, would be visible, which would contract according as the earth approached to the equinox, after which a like extent of its southern disk would be successively developed till the moment of the winter solstice. This spectator would therefore suppose that there existed in the earth a motion of libration.

(*k*) It may be objected to this explanation, that in consequence of the great rarity of the lunar atmosphere, no explosion would be visible; but in answer it is sufficient to observe, that there are several substances which develope during their ignition the oxygen gas, which is required in order that they may burn.

CHAPTER V.

(*l*) l and L denoting the heliocentric longitudes of the planet and earth, λ the geocentric longitude of the planet, we have

$$r.\cos.l - \rho.\cos.\lambda = \cos.L, \quad \text{and} \quad r.\sin.l - \rho.\sin.\lambda = \sin.L,$$

therefore,

$$\tan.\lambda = \frac{r.\sin.l - \sin.L}{r.\cos.l - \cos.L}, \text{ and } \frac{d\lambda}{\cos.^2\lambda} =$$

$$(r.\cos.l - \cos.L)\,(r.\cos.l\,dl - \cos.L\,dL) + (r.\sin.l - \sin.L)$$
$$(r.\sin.l.\,dl - \sin.L.\,dL) \div (r.\cos.L - \cos.L)^2$$

equal by concinnating to

$$\frac{\{r^2 - r.\cos.(L-l)\}.dl + \{1 - r.\cos.(L-l)\}.dL}{(r.\cos.l - \cos.L)^2};$$

but as the mean motions which are proportional to dL, dl, are inversely as the periodic times, we have $dL : dl :: r^{\frac{3}{2}} : 1$, unity denoting the radius of the earth's orbit, therefore $dL = dl.\,r^{\frac{3}{2}}$, $\therefore$ if $\frac{1}{r.\cos.l - \cos.L}$ be put equal to $\frac{P}{\cos.\lambda}$, we shall have $d\lambda = P^2.(r^2 + r^{\frac{3}{2}}) - (r + r^{\frac{5}{2}}).\cos.(L-l).\,dl$; in inferior conjunction or opposition $L - l = 0$, therefore $d\lambda = P^2.\,r.(r + r^{\frac{1}{2}} - 1 - r^{\frac{3}{2}}).\,dl = P^2.\,r.\,(r-1)\,(1 - r^{\frac{1}{2}}).\,dl$, which is always negative, hence the motion of the planet is always retrograde, in superior conjunction $L - l = 180$, therefore $\cos.L - l = -1$, hence λ must be positive, therefore the motion is direct; when $d\lambda = 0$, the planet appears stationary from the earth, and then we have $\cos.L - l = \frac{r + r^{\frac{1}{2}}}{1 + r^{\frac{3}{2}}}$. If m, m' represent the daily motions in longitude, we have $L = mt$, $l = m't$, and $L - l = (m - m')t$, t being the time when the longitude was the same, *i. e.* the time of syzygy, therefore as t in this case $= \frac{L-l}{m-m'}$ the planet will be retrograde while it describes $2m.\frac{L-l}{m-m'}$, and direct while it describes $360° - 2m.\frac{L-l}{m-m'}$,

hence it appears, that the greater the difference between m and m', the less the arc of retrogradation. The preceding investigation goes on the supposition that the orbits are circular, which is not the case, therefore it is that the arc of regression, and also the duration, are not always of the same magnitude.

(m) The illuminated portion of a planet varies as the versed sine of the exterior at the planet, *i. e.* as $1+\cos.\phi$, where ϕ is the angle at the planet, when ϕ is a maximum, *i. e.* when $\sin.\phi=\frac{1}{r}$ the planet is most gibbous, which is evidently in quadrature for a superior planet, in superior conjunction and opposition $\phi=0$, therefore the whole disk is illuminated; for an inferior planet, $\phi=180$ in inferior conjunction, hence in this position $1+\cos.\phi=0$, and the disk is invisible.

(m) M and m representing the angular velocities of the earth, t the time between two conjunctions, we have $t.(m-M)=360$, $m=\frac{360}{p}$, and $M=\frac{360}{P}$, therefore $\left(\frac{1}{p}\pm\frac{1}{P}\right).t=1$, and $t=\frac{Pp}{P\pm p}$.

(n) It is the parallax of Venus which is obtained by this method; however as its ratio to the parallax of the sun is known from having the ratio of the distances, which latter is given from the observed periods of the sun and Venus, we obtain the parallax of the sun; the transit of an inferior planet over the disk of the sun is a phenomenon of exactly the same kind as that of a solar eclipse, and may be calculated in precisely the same way. The parallax of the sun may be also inferred from theory.—*See* Book 4, Chap. 5, Vol. 2.

(o) It was originally proposed to observe the difference between the times of total ingress of Venus, as seen from two different places on the earth; this requires that the difference of longitudes of the two places should be known

accurately; besides it supposes that the spectators are either accurately, or very nearly in the plane of the orbit of Venus; to avoid this it was suggested, that by comparing the difference of duration of the transits, as seen from the different places, we might determine the parallax. From an approximate knowledge of the sun's parallax, we can compute the difference of duration at any place, compared with what it would be as seen from the centre of the earth. Hence, comparing the difference of duration at two distant places, at one of which the duration is shortened, and at the other lengthened, we get a double effect of parallax. It is, therefore, a matter of considerable importance to select places where the effects of the increase of the duration, or of its diminution, is greatest, and it is clear that with respect to the first, the duration is most lengthened when the commencement is near sun-set, and the end near to sun-rise; but in order to secure this it is evident, that the place must have a very considerable northern latitude; the duration would be evidently most shortened when the commencement was near sun-rise, and the termination near sun-set; hence, as the duration is only six hours, and as the time of the occurrence of the last transit was in June, it was necessary that the place should be to the south of the equator, where the days were then shorter than the nights; in places where the complement of latitude was less than the sun's declination, the sun would not set, consequently in such places the entire transit is visible, and the sun's elevation being then inconsiderable, the effect of the parallax would be very great; and also as Venus is depressed, the duration is increased.

(*p*) The law here adverted to is that which connects the periods and distances, namely, that the squares of the periods are as the cubes of the distances.

(*q*) Calling x the number of revolutions made by the earth, and y the number made by Venus in the interval between two conjunctions, we must have $xP=yP'$, P, P'

being the periods of the earth and planet, hence we must have $\frac{x}{y}=\frac{P'}{P}$, and by substituting for P′, P, we find, by means of the principle of continued fractions, that the numbers expressing this ratio are $\frac{8}{13}$, $\frac{235}{382}$, $\frac{717}{1139}$, &c. It does not necessarily follow that a transit will happen at these intervals, for it is likewise requisite that the least distance of the sun and Venus must be less than the sum of their semi-diameters, and as the nodes of Venus's orbit regrade, we cannot be ascertained of this without computation.

(*r*) The rotation of Mercury is not stated in the text; however Schroeter thought that certain periodical inequalities observed near the *horns* of his disk seemed to indicate a revolution in 24^h, $5'$, $30''$ on an axis, which coincided very nearly with the plane of his orbit. It was by a continued observation of the *horns* of Venus that he ascertained its rotation. The asperities of this planet, and the different situations of the shades which they project from the side opposed to the sun, change the form of the horns in the course of 23^h, $21,'$ $29''$; this can only be explained by the circumstance of its rotation, and that the horns resume always the same form at the end of a revolution. The compressions of Venus and Mercury ought not, if the time of their rotation be nearly the same as that of the earth, sensibly to differ from that of the earth, however the observed compression of Venus is nearly insensible.

Schroeter observed when the planet was dichotomized, that a bright spot moved very nearly in the line of the horns, hence he inferred, that the motion was very nearly perpendicular to the ecliptic; however, some uncertainty rests on this matter. Hence it appears, that since the mean length of a revolution is nearly the same for Mercury, Venus, and the Earth, there must be a much greater variation in the length of the days, and also in the seasons for Venus and

Mercury, as the inclination of their equator to their ecliptic, is considerably greater; indeed their torrid zone must embrace very nearly 150 or 180°; in fact, as the sun ranges to within 15° of one pole, the cold and darkness experienced at the other must be very great. It was to a mountain, situated near to the southern horn of Venus, that Schroeter directed his observations; strictly speaking, the line of the horns should be always a diameter, and those of a crescent should be very pointed; however, Schroeter remarked, that this was not always the case with respect to Venus, the horn of the northern extremity was always pointed, but the southern horn appeared sometimes obtuse, or blunted, which indicates the existence of a mountain, which covers a part with its shade.

To find the position of a planet when brightest, let k, r, and ρ denote the distances of the sun and earth, the sun and planet, the earth and planet, and χ the angle at the planet, the quantity of light received at the earth will vary as $\frac{1+\cos.\chi}{\rho^2}$, $\cos.\chi = \frac{\rho^2+r^2-k^2}{2r\rho}$, and $1+\cos.\chi = \frac{(r+\rho)^2-k^2}{2\rho r}$, consequently the quantity of light will vary as $\frac{(r+\rho)^2-k^2}{\rho^3}$, therefore differentiating this we obtain $0=\rho^2+4\rho r-(3k^2-r^2)$; and $\rho=-2r\pm\sqrt{3k^2+r^2}$, hence we obtain the value of $\cos.\chi$.

CHAPTER VI.

(*s*) THE motions of Mars are subject to more variations than those of any other planet, which circumstance induced Kepler to direct his observations more particularly

to this planet. The position when stationary, and also the duration of his direct and retrograde motion, is computed in the same manner as for an inferior planet. The cause of the differences which are observed in the quantity and duration of the retrogrations, arises from the ellipticity of the orbit.

(*t*) The brilliancy of a fixed star when approaching this planet was observed to become sensibly faint, hence it was inferred, that Mars was environed by a dense atmosphere, which was the cause of this faintness. Besides, from a continued observation of the spots, particularly two, which are near to the poles, there was observed a periodical increase and diminution, according as they are exposed to the action of the sun's rays in a more or less oblique manner; from this circumstance it has been conjectured, that they are like the collections about our polar regions of the earth.

(*u*) The inclination being very nearly the same as the inclination of the earth's axis to the ecliptic, the variations of seasons must be also nearly the same.

CHAPTER VII.

(*v*) The duration of Jupiter's rotation is the shortest, and his magnitude and mass are the greatest of any of the planets. This great rapidity of rotation may compensate for the greater weight which bodies experience at the surface of this planet, (*see* Vol. 2, Chap. 8, page 143); in fact, a point on the surface of Jupiter moves twenty-six times faster than a point on the earth's surface.

In consequence of the inclination being so inconsider-

able, it follows, that there is no great variety in the seasons.

(*w*) If the commencement and termination of an eclipse be accurately observed, then the middle of the eclipse is found, which is nearly the time when Jupiter is in opposition with respect to the satellite; let the time of another opposition, separated by a considerable interval from the first, be found in the same manner, calling τ this interval, and n the number of oppositions which have occurred in τ, we have T the time of a synodic revolution $=\frac{\tau}{n}$, hence, if P′ be the periodic time of Jupiter, we shall have P, the period of the satellite, $=\frac{P'T}{P'+T}$. *See* Notes to Chap. 9, Book 2.

It has been also inferred, from the circumstance of the greatest elongations of the satellites, when measured with a micrometer at their mean distances from the earth, being always the same, that the orbits are Q. P. circular, and it is in this manner that the distances are found in terms of the radius of Jupiter's equator; however, as in a comparison of a great number of observations, we must modify a little the laws of circular motion for the orbit of the third satellite; it follows, that this orbit is elliptical.—*See* Chap. 10, Book 2.

Calling Jupiter's geocentric longitude λ, l, the longitude of the satellite, as seen from Jupiter, and θ the longitude of the sun, the angle at the earth is equal to $\lambda - \odot$, that at Jupiter, $= l - \lambda$, and $r : 1 :: \sin.(\lambda - \odot) : \sin.(l - \lambda)$.

(*y*) From this circumstance of their alternately surpassing each other in splendour, it is probable that certain parts of their surface reflect more light than others, and then the epochs of the maximum or minimum of illumination ought to happen when the very same parts of the satellites are turned towards us; from a comparison of these returns with the positions of the satellites relatively to Jupiter

he ascertained that they always present the same face to this planet, hence he inferred, that they revolve on their axes in the time of their revolution about Jupiter.

Naming t, T the durations of the longest and shortest eclipse of the same satellite, and r the radius of Jupiter's equator, we have $T : t :: r : \frac{r.t}{T} = c$, half the chord of the arc described in the shortest eclipse, consequently d its distance from the centre of Jupiter $= r.\sqrt{1 - \frac{t^2}{T^2}}$; but $r : 360 :: T : L$, (a synodic revolution of the satellite,) $\therefore r = \frac{360°.T}{L}$, hence $d = \frac{T}{L} \cdot 360°.\sqrt{1 - \frac{t^2}{T^2}}$, and calling n the longitude of the satellite, and l that of Jupiter, we have in the right angled spherical triangle, of which the hypothenuse is $n-l$, and d a side about the right angle $\sin. d = \left(\sin. \frac{360° \sqrt{T^2 - t^2}}{L}\right) = \sin.(n-l). \sin. N$, (N being the inclination which consequently can be found). n is found by observing the position of Jupiter when the duration of the eclipse is the greatest possible, for the heliocentric longitude of Jupiter and of his node are in this case precisely the same.

CHAPTER VIII.

(z) The circular ring must always appear as an ellipse, as the eye of the spectator invariably looks at it obliquely, being never raised 90° above the plane of the ring, therefore the major axis of the ellipse is to the minor as radius to the sine of the angle ϕ, at which the line drawn from

the earth to the centre of the ring, is inclined to its plane, consequently if λ represent the geocentric longitude of Saturn, the longitude of the earth as seen from Saturn $=180+\lambda$, and if the longitude of the ring's node $=n$, B being the geocentric latitude of Saturn, and consequently $-$B the latitude of the earth as seen from the planet, we can, from knowing $180+\lambda-n$, B, and also v the inclination of the plane of the ring to the ecliptic, compute ϕ, and thus obtain the ratio of the axes of the ellipse $=\sin.\phi=\sin.\nu.\cos.\mathrm{B}.\sin.(n-\lambda)+\sin.\mathrm{B}\cos.v$, if $\sin.\phi=0$, i.e. if the earth is in the plane of the ring, we shall have $\sin.(n-\lambda)=\tan.\mathrm{B}.\cot.\nu$, in this case the thickness of the ring is turned towards us, which being inconsiderable is therefore invisible; this occurs twice during each revolution of Saturn, *i.e.* every fifteen years; if the plane of the ring passes through the sun it will disappear, because its thickness is then only illuminated; naming ϕ' the elevation of the sun above the plane of the ring, H h the heliocentric longitude and latitude of Saturn, we have $\sin.v'=\sin.\phi'.\cos.h.\sin.(\mathrm{H}-\mathrm{N})-\cos.v.\sin.h$, and therefore $\sin(\mathrm{H}-\mathrm{N})=\cot.v.\tan.h$, when $\phi'=0$, *i.e.* when the ring disappears. When ϕ, ϕ' have the same sign, the earth will see the illuminated part, and the ring will be visible; when they have contrary signs the ring will be invisible, for the ring will turn one of its faces towards the earth, and the other towards the sun; but as λ and H never differ by $5°$, which is described by Saturn in five months nearly, this difference of sign cannot last longer; it is in this interval that the phenomena of the appearances and disappearances occur, Saturn being near to his nodes, similar phenomena occur at the following node; if the ring disappears a short time before Saturn becomes stationary, the earth will meet it soon again, since Saturn becomes retrograde after the second occurrence, the ring will again become visible as $\sin.\phi$ and $\sin.\phi'$ then have the same sign, shortly after $\sin.\phi'$ vanishes, the plane of the ring passing through the sun, and as afterwards $\sin.\phi'$ changes

its sign, when the ring continues to be invisible until a short time after the planet becomes direct, when sin. ϕ vanishes, and consequently the plane of the ring passes through the earth, afterwards as sin. ϕ changes its sign, it will be the same as sin. ϕ', and consequently the ring will be visible for fifteen years; if when the plane of the ring passes through the sun, the angular distance of the earth from the ascending node of the ring, as seen from the sun, is greater than 90, and less than 180, there will be only one disappearance, which commences when the sun passes through the plane of the ring, and ends when the earth meets it, consequently it will last less than three months; if the preceding angle is > 180 and < 270, the earth meets the plane shortly before this plane passes through the sun, after this, sin. ϕ, sin. ϕ' will have the same sign, consequently the ring will be visible, consequently in this case as well as the preceding, the invisibility lasts three months; if this angle be > 270 and < 360, there will be two disappearances, namely, when the earth meets the plane of the ring a little before it passes through the sun, after this the earth again meets the plane of the ring, consequently there will be a second disappearance. If this angle be between 0 and 90, there are two disappearauces also, namely, when the earth meets the ring before opposition; secondly, when the ring passes through the sun after opposition, after the second reappearance, the ring bocomes visible for fifteen years. The most favourable circumstances for seeing the ring are when the plane passes through the sun and earth at the same time, the earth being in conjunction; then the earth is always on the illuminated side of the ring, which only ceases to be visible in consequence of the plane passing through the sun, if the plane passes through the earth and sun at the same time, the planet being in opposition, the circumstances for seeing the ring are the most unfavourable, in this case the ring is invisible nine months, four months before the passage of the plane through the sun, and

five months after. If $\sin.\phi$ be positive, we see the northern face of the ring; the semi-ellipse, which will be visible, will be below the centre of Saturn, and the otherhalf will be behind the planet; if $\sin.\phi$ be negative we see the half above the centre. The inclination of the ring to the ecliptic, or the angle v = the angle at the earth + the angle which a visual ray from the earth makes with the border of the ring, this last angle = 30°, the first angle = the geocentric latitude of Saturn + the angle which the minor semi-axis subtends.

$\text{Sin.}\,\phi=\frac{b}{a}$, therefore according as the earth ascends above the plane of the ring, the ellipse increases, when $b=a\sin.\phi=\frac{1}{2}$ diameter of Sáturn, its extremities coincides with those of its disk, in this case evidently $\sin.\phi=\frac{3}{7}$, if $n-\lambda=90$, $\frac{b}{a}=\sin.\phi=\sin.v.\cos.g-\cos.v.\sin.g=\sin.(v-g)$, therefore $\phi=v-g$, and since $n=90+\lambda$, we have the place of the ascending node. As the phenomena of the disappearances recur after a complete revolution of Saturn, it follows, that these two positions of the ring always correspond to the same points of the orbit of Saturn, and consequently the plane remains always parallel to itself, therefore its inclination to the ecliptic is invariable, or if we substitute for $\tan.h$ its value $\tan.I'.\sin.(H-N')$, I' and N' being the inclination and longitude of the node of the orbit of Saturn, we have

$$\sin.(H-N)=\cot.v.\tan.v'.\sin.(H-N'),$$

therefore,

$$\tan.v'.\cot.v=\frac{\sin.(H-N)}{\sin.(H-N')}=\frac{\sin.H.\cos.N-\cos.H.\sin.N}{\sin.H.\cos.N'-\cos.H.\sin.N'}$$

$$=\frac{\tan.H.\cos.N-\sin.N}{\tan.H.\cos.N'-\sin.N'},$$

therefore,

$$\tan. H = \frac{\tan. v'. \cot. v. \sin. N' - \sin. N}{\tan. v'. \cot. v. \cos. N' - \cos. N};$$

hence we find H, which is very nearly constant. When the plane of the ring passes through the sun, the heliocentric longitude of Saturn on the orbit $= N$, ∵ the place of the nodes of the ring on the orbit is determined, or *vice versa*, which is found to be the same always; let $N_{,}$ be this longitude reduced to the ecliptic, the angle at the sun between rad. of earth and curtate distance of Saturn $= N_{,} + 180 - \odot$, the angle at the earth subtended by curtate distance $= \odot - z$, rad. of earth $= a$, we have curtate distance $= \frac{a. \sin. (\odot - z)}{\sin. (z - N_{,})}$, and radius vector of Saturn $= \frac{\text{custate distance}}{\cos. \phi}$. ($z =$ the geocentric longitude of Saturn).

The apparent headth of the ring is equal to the distance of its interior border from the surface of Saturn, as is indeed evident from what has been already observed, and it revolves in a time equal to the periodic time of a satellite whose distance from Saturn would be the same as that of the ring.

CHAPTER IX.

(*a*) In determining the elements of the planetary orbits, a great number of observations is supposed to be made about the time of opposition or conjunction, and also the periodic times of the planets are supposed to be known; but as this last element is most accurately determined by means

of a great number of complete revolutions, and as the motion of this planet is so slow as to preclude the possibility of observing more than one opposition in eighty years, a considerable time must elapse before the elements of Uranus can be known with the same accuracy as those of the other planets. However, as will be shewn in the third Chapter of the second Book, the very extreme slowness of the observations enables us to make a tolerably accurate approximation to a knowledge of the elements.

In the consequence of the *q. p.* perpendicularity of the planes of the orbits of the satellites of Uranus to that of their primary, they must experience considerable disturbance from the action of the sun; indeed the investigation of the sun's action would be a new case in the problem of the three bodies, for in general the inclinations are assumed to be inconsiderable.—*See* Vol. ii. p. 51.

CHAPTER X.

(*b*) These planets are so small that they belong to that class of stars which are termed telescopic, the volume of all the four taken together does not surpass the magnitude of the moon, therefore, though nature has elevated them above the rank of satellites, as far as their magnitude is concerned, they are below these bodies. These circumstances of their extreme smallness, and of their being at the same distance very nearly from the sun, have induced philosophers to think that they are the fragments of one planet divided into parts; indeed an explosion with a velocity twenty times greater than that of a cannon ball, would be sufficient to make these detached fragments de-

scribe orbits similar to those described by these planets; such an hypothesis explains why the excentricities and inclinations of these planets are so considerable, and also why they are moved in such various directions, and with such different velocities.—*See* Vol. ii. p. 250.

The elements of these planets cannot be known with the same precision as those of the other planets, for as not more than five revolutions of them have been observed, their periodic time, a most important element, cannot be determined with great accuracy. The best method for determining the elements of these stars is that given by M. Gauss in his *Theoria motus corporum cœlestium;* but when their proximity to Jupiter, the perturbations which result from their mutual attractions, and their great eccentricities and inclinations, are taken into account we cannot expect to have as yet a very accurate knowledge of these elements.

CHAPTER XI.

An epicycle is a curve produced by the combination of two circular motions. The circles described by the centres were called deferents. The epicycle of a superior planet was supposed to be described in the time between two conjunctions or oppositions. The epicycle of an inferior was described in the time between two inferior conjunctions. The deferent of the superior planets were supposed to be described in the time of a planet's revolution about the sun; those of inferior planets in the time of the earth's revolution. In a position of an inferior planet on the Ptolemaic system, if lines be drawn from the planet to the earth and centre of the deferent, the angle at this centre

will be equal to the angle which an inferior planet has gained on the earth since last inferior conjunction, hence if the rad. of the deferent is to the rad. of the epicycle, as the distance of the sun from earth, to the distance of the sun from planet, the angle at the earth is equal to the angle of elongation in the true system, and if the rad. of deferent be assumed equal to the distance of the sun from the earth, (which we are permitted to do on Ptolomy's system), we have then the inferior planets moving about the sun, which is itself carried in a year about the earth; in like manner, if lines be drawn from the place of superior planet to earth, and to centre of deferent, the angle at the centre will correspond to the angle gained by the earth on the planet, and if those distances are proportional to the distances of the earth and planet from the sun, the angle made by lines drawn from earth to sun and planet, will be equal to the elongation, for the rad. of the epicycle may be shewn to be parallel to the moveable rad. of the sun; it is evident also, that if the rad. of the deferent be equal to the distance of the planet from sun, the rad. of the epicycle is equal the distance of the sun from the earth.

CHAPTER XIII.

(*a*) Considering the great perfection of Astronomical instruments, and the precision with which observations have been made, it is supposed that if the parallax was equal to 3″ of the decimal division of the circle, or 1″ of the sexagesimal, it might be observed; if the parallax was equal to 9′ 1″, the diameter of the earth's orbit would hardly subtend an angle equal to the thickness of a

spider's thread at the star. Various methods have been suggested for determining the distances of the fixed stars, of which the most successful appears to be that which was first suggested by Galileo, and subsequently improved on by Herschell, of which the principle consists in determining the angle, or variations in the angle, which two stars very near to each other appear to subtend at opposite seasons of the year.—*See* Philosophical Transactions for the year 1794.

Another method was from the consideration of the quantity of light in the stars compared with the light of the sun; in this way M. Mitchell concluded, that the parallax of a star of the second magnitude is not more than the $\frac{1}{3}$th of a second, and of a star of the fifth or sixth magnitude not more than $\frac{1}{20}$ or $\frac{1}{30}$th of a second. The attempts to discover the parallax of the stars by direct observations, have not been attended with any success previously to the time of Doctor Brinkley, Professor in Trinity College, Dublin. His observations which were made with the greatest care, seem to indicate the existence of parallax in a Lyræ amounting to 2″, 52. *See* Philosophical Transactions for the years 1812 and 1813.

(*b*) On the hypothesis, that Sirius was of the same magnitude as the sun, Huyghens found by diminishing the aperture of a telescope, so that the sun when seen through it might appear of the same apparent magnitude as *Sirius*, that the diameter of the sun was diminished in the ratio of 1 : 27664, hence *Sirius* is 27664 times more distant than the sun.

The smallest apparent diameter of an opaque body which is visible is about 40″, but if the body be luminous *per se*, the limit of visibility will be so much less as the light of the body is stronger, and as the stars with a diameter less than 1″ have a splendour so great that some of them are visible immediately after sun-set, there cannot be

any doubt but that they have a light of their own like the sun; their extreme smallness is proved from their scintillating, which shews that the least molecule floating in the air is sufficient to intercept their light.

When a fixed star is eclipsed by the moon it ought to disappear by degrees, if it had a sensible apparent diameter, conformably to the moon's mean motion, which is such that it describes its apparent diameter, which is about 30′ in an hour, consequently in two seconds of time it ought to describe one second of space.

(*c*) A third explanation of these phenomena has been suggested; this supposes that the figures of these stars is very compressed, which makes them to appear much less flattened in some aspects than in others.

(*d*) The milky way environs the sphere very nearly in the plane of a great circle, which by half of its breadth intersects the equator at the 100th and 277th degree, its inclination to the equator is equal to 60°, the breadth is from 9 to 18 degrees; it is narrowest near the poles of the equator, between the constellations Cassiopea and Perseus, and its greatest breadth is in the plane of the equator. The milky way is divided into several departments, by a space void of stars, in the middle of the breadth, chiefly from 254° of right ascension, and 40 of south declination to 310 of right ascension, and 45° of northern declination. Hershel could distinguish the stars of which this milky way was composed, which were so near to each other, that in telescopes of inferior magnifying power their light was confounded; according as the direction of the telescope deviated from the milky way, the number of these stars diminished. Having counted the stars in different parts of this way, he found that on a medium estimate, a segment 15° long, and two degrees wide, contained 50,000 stars, of sufficient magnitude to be distinguished through his powerful telescope; ∴ on the

supposition that the breadth of the milky way is 14°; it follows, that it contains more than eight millions of stars, without reckoning those, which even with this great telescope cannot be distinguished; with respect to the arrangement and nature of the stars which constitute the milky way, some observations will be suggested in the Notes to Chap. VI. of the 2nd Volume.

(*e*) The declination of an object is best obtained by observing its distance when on the meridian from the horizon or zenith, for this distance added or subtracted from the distance of the zenith from the pole, gives the distance of the object from the pole, and consequently the declination; if the object has apparent magnitude, the altitudes or zenith distances of its upper and lower limbs should be observed, and then half their sum should be taken as the altitude of the centre; this method requires that the exact zenith distance, corrected by refraction and parallax should be known, which is best obtained in the manner indicated in the notes to the first chapter.—*See* also Notes, to Chap. XIV. If the star does not exist in the meridian, then in order to determine the declination, it is necessary to know the zenith distance of the star, that of the pole and either the azimuth or hour angle from noon. Indeed, of the five preceding quantites any three being given, the other two may be found by the solution of a spherical triangle. This general problem contains twenty different cases, of which the most useful are given in the Treatises of Astronomy. However, there is an obvious advantage in determining the declination by means of an observation made in the meridian, for in this case parallax and refraction only affect the declination, but do not at all alter the right ascension.

With respect to the right ascension, its determination is more difficult than the declination, as the first point of Aries, from which it is reckoned, is not fixed in the hea-

vens. The difference of the right ascension of two stars is obtained by observing the time intervening between their passages over the meridian; this converted into time at the rate of 15° for an hour, gives the difference. Hence, as this difference is easily observed, if we had the right ascension of some one star, that of all others might be determined; the method which Flamstead proposed was as follows:

He noted when the sun had equal declinations, some time after the vernal and before the autumnal equinox; in these positions the distances of the sun from the respective equinoxes must be the same, call this distance E, and let D, $D'+p$, represent the differences of right ascensions of the sun and some star in these two positions, then we have $D+D'+p+2E=180$, hence we obtain E, and consequently the right ascension of the star; p is the correction to be made to the right ascension, in consequence of precession and displacement of ecliptic, which will be afterwards noted.

It is easy to compute the angular distance of two stars, of which we know the right ascensions and declinations, for if d, d', represent the polar distances of the stars, and Δ the angle made by d, d' at the poles, which is measured by their difference of right ascension, and D the arc of a great circle which measures their angular distance, we have by the formulæ of spherical trigonometry, $\cos. D=\sin. d. \sin. d'. \cos. \Delta+\cos. d. \cos. d'$. Let λ represent the longitude, β the latitude, ρ the right ascension, δ the declination of a star; p its angle of position; π the arc of a great circle intercepted between the star and equinoxial point; ϕ the angle contained between this arc and equator, and ε the obliquity of ecliptic; then $\cos. \phi=\tan. \rho. \cot^{nt}. \pi$; $\cos. (\phi-\varepsilon)=\text{tang.} \lambda \cot^{nt}. \pi$; $\sin. \delta=\sin. \phi. \sin \pi$; $\sin. \beta=\sin. (\phi-\varepsilon). \sin. \pi$. Hence we obtain $\tan. \lambda=\dfrac{\cos. (\phi-\varepsilon). \tan. \rho}{\cos. \phi}$ and

$\sin. \beta = \sin. \frac{(\phi - \varepsilon)}{\sin. \phi} \sin. \delta$; hence we can obtain λ and β when ϕ is known.

The reverse formulæ for finding ρ and δ from knowing λ and β, may be obtained by merely changing β into δ, and λ into ρ, and by making ε negative; θ representing the arc of the circle of declination passing through the star, intercepted between equator and ecliptic; we have $\sin \theta = \tan. \rho. \cot^{nt}. \nu$, and $\cos. (\delta - \theta) = \cot. p. \cot \nu$, $\therefore$ $\cot^{nt}. p = \frac{\cos. (\delta - \theta) \tan. \rho}{\sin. \theta}$ (note $\nu =$ the angle at which circle of declination passing through the star, is inclined to ecliptic) $\cos. \delta. \cos. \rho. = \cos. \lambda. \cos. \beta$; $\tan. \rho = \frac{\cos. (\phi + \varepsilon). \tan \lambda}{\cos. \phi.}$ and $\sin. \delta = \frac{\sin. \beta. \sin. (\phi + \varepsilon)}{\sin. \phi}$; $\tan. \rho$, may become negative in several cases, in the first quadrant, if $\sin. \lambda$ is less than $\tan. \varepsilon. \tan. \beta$, for by substituting for ϕ, the preceding value of $\tan. \rho = \frac{\cos. \varepsilon. \sin. \lambda}{\cos. \lambda} - \frac{\sin. \varepsilon. \tan. \beta}{\cos. \lambda}$ this may happen when λ is small and β great, *i. e.* if the star is in the circle of latitude near to the pole of the ecliptic; in the 2nd quadrant $\tan. \rho$ is negative, unless that $\sin. \lambda$ is less than $\tan. \varepsilon. \tan. \beta$; in the 3rd quadrant $\tan. \rho$ is positive, and in the 4th quadrant — $\therefore$, except when $\sin. \lambda$ is less than $\tan. \varepsilon. \tan. \beta$; ρ is always in the same quadrant as λ; ρ is negative or in the 4th quarant if $\lambda = o$; unless $\beta =$ either o, or is —; in the first case $\rho = o$, *i. e.* the star is in the equinoxial point; in the second case it is in the first quadrant; if $\lambda = 90$, and $\tan. \beta >$ than $\sin \lambda. \cot. \varepsilon = \cot^{nt}. \varepsilon$, *i. e.* if β is greater than 66° 32′, in this case $\tan. \rho = - \infty$. and the star is in the solstitial colure between the two poles.

(*e*) As the distance between the pole of the equator and the pole of the ecliptic = the obliquity of the ecliptic which is very nearly constant, it follows that the axis of the equator describes, in consequence of this precession, a cone about the pole of the ecliptic. In order to obtain the variation in right

ascension and declination; supposing β and ε constant, we shall have by differentiating $\frac{d.\,\delta}{d.\,\lambda} = \frac{\sin.\,\varepsilon.\,\cos.\,\beta.\,\cos.\,\lambda}{\cos.\,\delta.}$

and $\frac{d\,\rho}{d\,\lambda} = \cos.^2\,\rho\left(\frac{\cos.\,\varepsilon - \sin.\,\varepsilon.\,\tan.\,\beta.\,\sin.\,\lambda}{\cos.^2\,\lambda}\right)$ and as we have from comparing the two values of p obtained by substituting,

$$\cos.\,\rho\left(\frac{\cos.\,\varepsilon - \sin.\,\varepsilon.\,\tan.\,\beta.\,\sin.\,\lambda}{\cos.\,\lambda}\right)$$
$$= \frac{\cos.\,\varepsilon\,\cos.\,\delta + \sin.\,\varepsilon\,\sin.\,\delta.\,\sin.\,\rho}{\cos.\,\beta}$$

by substituting this value and from the equation $\cos.\,\rho.\,\cos.\,\lambda = \cos.\,\lambda.\,\cos.\,\beta$, these will assume the form. $d.\,\delta = d\,\lambda.\,\sin\,\varepsilon.\,\cos\,\rho$; $d\,\rho = d\,\lambda\,(\cos.\,\varepsilon + \sin.\,\varepsilon.\,\tan.\,\delta.\,\sin.\,\rho.)$ Note, as the equinoxes regrade uniformly, $d\,\lambda$ is constant, and it appears that the right ascension cannot diminish except when the star is in the southern hemisphere, or when it is in the 3rd or 4th quadrants, $\tan.\,\delta.\,\sin.\,\rho.$ being greater than $\cot.\,\varepsilon$; and as near to the pole $\tan.\,\delta$ approaches to ∞, the variation of right ascension may become then indefinitely great.

The preceding formulæ are sufficiently exact, when the effects of precession are computed for an interval which is near to the epoch, for which we have determined the arguments ε, δ, ρ. But as ε changes continually within certain limits as shall be observed in Chapter XIII, Vol ii, and as $d\lambda$ likewise, is not always the same, the preceding expressions are only correct for a short interval of time.

Bradley, in his endeavour to ascertain whether the parallax of the fixed stars was of a sensible quantity, observed that for the space of nine years the declination of the stars increased, and that it diminished by the same quantity the nine following years; so that all was re-established after eighteen years. He likewise observed, that the greatest difference of declination was 18″, and that the latitude was not affected; hence he inferred, that the pole of the equator approached the pole of the eclip-

tic for the first nine years, and that it receded from it by the same quantity the following nine years. He observed, likewise, that this motion was connected with an irregularity of the precession of the equinoxes, which obeyed precisely the same period; hence it follows that the motion of the poles of the equator, does not take place in the solstitial colure, or in other words, that the poles neither describe right lines nor the arc of a great circle of the sphere, but a curve or small circle intersecting the solstitial colure; ∵ as the true motion of the pole takes place in the periphery of an ellipse of which the centre retrogrades on the periphery of the circle described by the mean place of the pole, its locus will be a species of epicycle. In the superior part the direction of the motion of the pole is the same as that of the epicycle, ∵ the actual motion being quicker than the mean motion, the true pole precedes the mean; it is the contrary in the lower part of the ellipse, and as the mean motion is considerably greater than the motion in the ellipse, it predominates over it; ∵ the motion in the epicycle is still retrograde. From a comparison of observations of the nutation with the nodes of the moon, it appears that the right ascension of the true pole, reckoned from the mean pole precedes by 90°, the longitude of the ascending node of the moon; *i. e.* $\rho = 90^\circ + \Omega$. then $d\,\varepsilon$ the variation of obliquity $= q.\ p$, the cosine of Ω to a radius $= 9''$, 65, *i. e.* $d\,\varepsilon = 9''$, 65. cos. Ω. To determine the variation in longitude, it is to be remarked that the angle formed by lines drawn to the true and mean poles of the equator, from the pole of the ecliptic = distance of true pole from the axis major, divided by sine of the distance between poles of the equator and ecliptic $= \dfrac{7'',\ 17,\ \sin.\ \Omega}{\sin.\ \varepsilon}$, see notes to Chap. XIII, Vol, 2, in order to determine the effects of nutation on the right ascension and declination,

naming δ' and ρ' the right ascensions and declination when the longitude becomes $\lambda + d\lambda$, and the obliquity becomes $\varepsilon + d\varepsilon$, then by formula given in page 341; we have,

$$\sin. \delta' = \sin. (\varepsilon + d\varepsilon). \cos. \beta \sin. (\lambda + d\lambda) + \cos. (\varepsilon + d\varepsilon) \sin. \beta$$

$$\tan. \rho' = -\tan. \beta. \frac{\sin. (\varepsilon + d\varepsilon) + \sin. (\lambda + d\lambda). \cos. (\varepsilon + d\varepsilon),}{\cos. (\lambda + d\lambda)}$$

then by omitting all terms after the first we obtain, $\delta' = d + d\lambda. \sin. \varepsilon. \cos. \rho + d\varepsilon . \sin. \rho$; $\rho' = \rho + d\lambda. (\cos. \varepsilon + \sin. \varepsilon. \sin. \rho. \tan. \delta) - d\varepsilon. \cos. \rho. \tan. \delta$; substituting for $d\lambda$, $d\varepsilon$, their values previously found, and making $9'', 65 = h$; $18'', 0\,3. \sin. \varepsilon = g$, we obtain the nutation in right ascension or the value of $\rho' - \rho'$, which is the same thing, $= - g. \sin. \Omega \cot^{nt}. \varepsilon - \tan. \delta. (h. \cos. \Omega. \cos. \rho + g \sin. \Omega \sin. \rho)$ the first term being independent of the stars place, is the same for them all; assuming $h. \tan. \delta. \cos. \rho = g. (\cot. \varepsilon + \sin \rho. \tan. \delta). \tan. B'$, then the nutation in right ascension $= - g. (\cot. \varepsilon + \sin. \rho. \tan. \delta. \sin. (B' + \Omega,)$ hence it is easy to perceive that for the same star the nutation in right ascension is a max^{m}, when $\Omega + B' = 90$; by making the substitutions already indicated, the nutation in declination or north polar distance, which is the same thing, becomes $- g. \cos. \rho. (\sin. \Omega - \frac{h}{g}. \tan. \rho. \cos. \Omega)$, let $- \frac{h}{g}. \tan. \rho = \tan. B$, and then $- \frac{g. \cos. \rho}{\cos. B} . \sin. (B + \Omega)$ $=$ the nutation in north polar distance; it is easy to perceive that this becomes a max^{m}, when $\Omega + B = 90°$.

Beside the nutation just examined, the pole of the equator is subject to a similar inequality arising from the disturbing action of the sun, it is much feebler than that of the moon, however, it is not altogether insensible, and is always introduced in the tables. In consequence of this action the true pole describes a circle about the mean

pole *according* to the order of signs; its period is half a year, and the true pole is always 90 before the sun; indeed, if extreme accuracy was required, it is theoretically true that in the course of half a month, the pole is disturbed from the inequality in the moon's action; however, this last is altogether insensible; now as the true pole would, in consequence of *each* of these actions, if they obtained separately, combined with the motion of the pole arising from precession, describe an epicycloid, the curve actually described, will be that which results from the combined action of all these motions; however, as they are separately extremely small, if we estimate the effect of each by itself, and then take the sum, the total effect may be considered very nearly as = to the sum of all the partial results.

With respect to the aberration of light, which is the third correction to be applied in order to obtain the true place of a star, *see* Notes to Chapter II. Book II.

Besides the three *apparent* motions of the fixed stars, which are adverted to in this chapter, namely the precession, the nutation, and the abberration, there is a fourth, which though obscurely indicated by observation, is completely established by theory, namely the diminution of the obliquity of the ecliptic. *See* Notes to Chapter XIII. Volume 2nd.

CHAPTER XIII.

(f) In order to a clearer understanding of the articles treated of in the text, it will be necessary to establish a few principles relating to the radius of curvature and expressions for a degree of the meridian, &c., for this

purpose let a, b denote the semi-axes of an ellipse, p the principal semi-parameter $=\frac{b^2}{a}$, n the normal and ρ the radius of meridional curvature, λ the latitude, $x\ y$ the co-ordinates of any point, and s its subnormal; then as $n^2 = y^2 + s^2$, and as $x = \frac{a^2}{b^2}\cdot s$, and as $y^2 = \frac{b^2}{a^2}\cdot(a^2 - x^2)$, we shall obtain by observing that $n \sin.\lambda = y$, $n.\cos\lambda = s$, $n.^2 \sin.^2\lambda = \frac{b^2}{a^2}\cdot\left(a^2 - \frac{a^4}{b^4}\cdot n^2\cos.^2\lambda\right)$ $\therefore$ by concinating, $n^2.$

$$(b^2.\sin.^2\lambda + a^2.\cos.^2\lambda) = b^4, \text{ and}$$

$$n = \frac{b^2}{\sqrt{a^2.\cos.^2\lambda + b^2.\sin.^2\lambda}};\ \text{and as } \rho = \frac{n^3}{p^2} = \frac{n^3.\,a^2}{b^4}$$

by substituting we obtain $\rho = \frac{a^2.\,b.^2}{(a^2.\cos.^2\lambda + b^2.\sin.^2\lambda)^{\frac{3}{2}}}$

hence we obtain $D = \frac{a^2.\,b^2.}{m\,(a^2.\cos.^2\lambda + b^2.\sin.^2\lambda)^{\frac{3}{2}}}$ (m expressing the number of degrees in an arc = to the radius.) Making $b = a - c$, and neglecting the square and higher powers of c; we obtain $m\,D = (a^4 - 2a^3.\,c).\ (a^2 - 2\,a\,c.\,\sin.^2\lambda)^{-\frac{3}{2}} = a\left(1 - \frac{2c}{a} + \frac{3c}{a}.\ \sin.^2\lambda\right) = a\left(1 - \frac{c}{2\,a} - \frac{3\,c}{2\,a}.\cos.2\lambda\right)$; $\because$ at the equator $m\,D = a - 2\,c$, and at the pole $m\,D = a + c$, at the parallel 45, $m\,D$ is an arithmetic mean between $m\,D$ at the equator, and $m\,D$ at the poles, for at $45°$ $m\,D = a - \frac{c}{2}$; If D' be a degree to the latitude

λ', we have $m\,D' = a - \frac{c}{2} - \frac{3\,c}{2}.\cos.2\lambda'$; hence

$$c = \frac{2\,m\,(D' - D)}{3.\,(\cos.2\lambda - \cos.2\lambda')} \text{ and } \frac{c}{a} =$$

$\frac{2\,(D' - D)}{3\,D.\cos.2\lambda - \cos.2\lambda')}$; these expressions may be reduced respectively into $c = \frac{2\,m\cdot(D' - D)}{3.\sin.(\lambda + \lambda')\sin.(\lambda - \lambda)}$; $\frac{c}{a}$

$= \frac{2(D'-D)}{3 D. \sin.(\lambda+\lambda) \sin.(\lambda'-\lambda)}$, ∵ at the equator $c = \frac{m(D'-D)}{3. \sin.^2\lambda}$ hence the increment of the degree at any latitude λ', above the degree at the equator is as $\sin^2\lambda'$; likewise as $D'-D \backsim \sin.(\lambda'+\lambda) \sin.(\lambda'-\lambda)$ if D' D are two contiguous degrees, so that, $\lambda'=\lambda+1°$; then $D'-D= \frac{3c}{m} \sin.(2\lambda+1°). \sin. 1°$; ∵ as the difference of contiguous degrees is ∷ l to $\sin.(2\lambda+1)$ it is a maximum when $2\lambda+1=90$, *i. e.* when the middle latitude is 45°. The semi-diameter² to any latitude $\lambda = r^2 = x^2 + y^2 = n^2 \sin.^2\lambda + \frac{a^4}{b^4}. n^2 \cos.^2\lambda \therefore r = n. \frac{\sqrt{b^4. \sin.^2\lambda + a^4 \cos.^2\lambda}}{b^2} = \frac{\sqrt{a^4 - 4a^3.c \sin.^2\lambda,}}{a^2-2ac}$ and by expanding this expression and neglecting c^2, we obtain $r = a\,(1 - \frac{c}{a} \sin.^2\lambda.)$

The circumference of the elliptic meridian may be found by multiplying the mean degree, *i. e.* the degree in the parallel of 45° by 360°. By the series expressing the rectification of the ellipse, it may be found still more accurately.

In an ellipsoid of revolution, the normal terminated in the minor axis is equal to ρ', the rad. of curvature of a degree perpendicular to the meridian, for as in this hypothesis the direction of gravity always passes through the axis af the earth, the direction of a plumb line which is perpendicular to the meridian, and indefinitely near to it on the east and west sides, will intersect the axis in the same point, which point is ∵ the centre of curvature of the arc; as this normal is greater than the rad. of meridional curvature, a degree perpendicular to the meridian is greater than a degree of the meridian; $\rho' =$

$$\frac{a^2}{\sqrt{a^2 \cos.^2\lambda + b^2 \sin.^2\lambda}}$$

Now,

$$x = \frac{a^2 \cos.\lambda}{\sqrt{a^2.\cos.^2\lambda + b^2.\sin.^2\lambda}}\ y = \frac{b^2.\sin.\lambda}{\sqrt{a^2.\cos.^2\lambda + b^2.\sin.^2\lambda}}$$

$\therefore$ as $r^2 = x^2 + y^2$ we obtain by substituting e^2 for $\frac{a^2-b^2}{a^2}$,

$r^1 = a\left(1 \frac{-e^2.(1-e^2).\sin.^2\lambda}{1-e^2.\sin.^2\lambda.}\right)^{\frac{1}{2}}$; naming h the angle at the centre between r and a, we have $\tan. h = \frac{b^2}{a^2}.\ \tan.\lambda$, if l represent the angle between a and a line drawn to the extremity of the produced ordinate, we have

$$\tan. l = \frac{b}{a}.\ \tan.\lambda,\ \because \sin.^2 l = \frac{\frac{b^2}{a^2}.\tan.^2\lambda}{1+\frac{b^2}{a^2}.\tan.^2\lambda},$$

hence by putting $1-e^2$ for $\frac{b^2}{a^2}$ we obtain $\frac{(1-e^2).\ \sin.^2\lambda}{1-e^2.\ \sin.^2\lambda}$

$= \sin.^2 l$, and $\therefore r = (1-e^2 \sin.^2 l)$, hence as l differs very little from λ, it follows as before that the increments of the rad. are very nearly as the squares of the sines of λ. π the angle between n and $r = \lambda - h$, $\therefore$ substituting for $\tan. h$ its value $\frac{b^2}{a^2}\tan.\lambda$, we obtain $\tan.\pi = \frac{\tan.\lambda - \frac{b^2}{a^2}\tan.\lambda}{1+\frac{b^2}{a^2}\tan.^2\lambda}$

$= \frac{(a^2-b^2)\tan.\lambda}{(a^2+b^2)\tan.\lambda}$; likewise it follows that

as we have always $\tan.\lambda + \tan. h : \tan.\lambda - \tan. h :: \sin.(\lambda+h) : \sin.(\lambda-h) :: a^2+b^2 : a^2-b^2$; it may be shewn that $\pi = \lambda - h$ is a max^m when $(\lambda+h) = 90°$; it is evident also from other considerations that the point where the angle between the r and n is a max^m, must be at the extremity of the equal conjugate diameters: if the

value of x, which is given above, be differentiated, we obtain after all reductions; $dx = = -a\,\frac{(1-e^2)\sin.\lambda\, d.\lambda}{(1-e^2\sin.^2\lambda)^{\frac{3}{2}}}$;

now $ds = -dx\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = -dx.\ \cos^{nt}\lambda$

$\therefore = a\,\frac{(1-e^2)\,d\,\lambda}{(1-e^2\sin.^2\lambda)^{\frac{3}{2}}}$; hence we derive $\rho = \frac{ds}{d\lambda}$, as

$$m\Delta = \frac{a^2}{\sqrt{a^2\cos.^2\lambda+b^2\sin.^2\lambda}} = a^2.\ (a^2-2\,a\,c.\sin.^2\lambda)-^{\frac{1}{2}}$$

$=a+c\sin.^2\lambda$; $m.(\Delta-\mathrm{D})=2\,c.\cos.^2\lambda$; hence we can determine c, a, &c. Δ representing the same as before, a degree of longitude $=\Delta\cos.\lambda$. If R denotes the rad. of curvature of any sect., perpendicular to the tangent plane, at the earth's surface, it would not be difficult to show that it was equal to $\frac{\rho\,\rho'}{\rho.\sin.^2\theta+\rho'\cos.^2\theta}$, θ being the angle which the cutting plane makes with the meridian; hence it follows that when θ is 45°, R is an harmonic mean between ρ and ρ'.—See *Puissant*, tom. I. p. 238.

It follows from the expression cos. ρ, that a degree of longitude at the equator is the first of two mean proportionals between first and last degrees of latitude, for D at the equator is to D at the poles, as $a^3 : b^3$, the general ratio being that of

$$(a^2\sin.^2\lambda+b^2\cos.^2\lambda)^{\frac{3}{2}} : (a^2\sin.^2\lambda'+b^2.\cos.^2\lambda')^{\frac{3}{2}}$$

The compression is obtained more accurately by comparing a meridional degree with a degree of a perpendicular to the meridian, than from a comparison of two meridional degrees with each other.—See *Puissant*.

The following is a brief outline of the method for determining the length of any arch of the meridian: two points are assumed nearly at the distance of the required arch, these two points are then connected by a series of triangles, the angles of which are determined by means of stations taken on the tops of hills, or other elevated posi-

tions; the angles of the triangles and also the azimuths of the sides, at the points where the series commences and ends, are to be measured. By this means the species of all these triangles are given, and also the bearings of their sides, with respect to the meridian of the first station. The lengths of the sides of the triangles are known by measuring a base on a level ground, and connecting it with the sides of one of the triangles. In these computations the process is on the supposition that the triangles are plane; however the error from this hypothesis is corrected by knowing the spherical excess which is given from knowing the area.—See *Puissant's Geodesique*, tom. I. p. 223.

($g^{\prime}$) In like manner the terrestrial equator may be defined to be the plane, formed by all the points of the terrestrial surface, the verticals of which are parallel to the plane of the celestial equator, or which is the same thing, which are perpendicular to the axis of rotation of the heavenly sphere; consequently unless the earth be a solid of revolution, the terrestrial equator will be a curve of double curvature; if it be a solid of revolution, the terrestrial equator is a great circle of the sphere.—(*sec* p. 102.) In like manner, the poles of the earth are those points of its surface, whose verticals are parallel to the axis of rotation; so that these points are not necessarily diametrically opposed to each other, except the earth be a solid of revolution. However, though when the earth is not a solid of revolution, neither the equator nor meridians are plane curvės, still the corresponding celestial equator and celestial meridians may be considered as great circles, for the verticals when indefinitely prolonged may be conceived as terminating in the celestial sphere, in different points of the same great circle.—See *Puissannt*, tom. II. Book 6th.

Conformably to the above definitions, the terrestrial parallels will be formed by points, of which the verticals

meet the celestial sphere under the same parallel, so that all points of the same parallel will have the same stars in the zenith; however, unless the earth be a solid of revolution these points will not form a circle, or even exist in the same plane. The latitude of all the points of these parallels is the same.—(*See* p. 111.) N. B. It is evident that the length of degrees of the terrestrial parallels decrease in proceeding from the equator to the pole, in the ratio of the cosine of latitude. From some measurements made by Biot and Arrago, it would appear that the parallel to the equator at the southern extremity of the meridian measured by them, is sensibly elliptic.

(*h*) ds denoting the first side of this line, ds' the second side, &c. These sides may be considered as equal, at least if quantities indefinitely small of the third and higher orders be neglected, for let i denote the angle which the prolongation af the first side (which is evidently equal to the first side) makes with the second, (i being a quantity indefinitely small of the first order,) then as the prolongation of the first side is evidently equal to it, we have $ds' = ds. \cos. i = ds - \frac{ds. i^2}{2}, = ds$, as ds is of the same order as i; hence it follows that in a geodesique line its differential is constant, likewise the normal comprised between the prolongation of the first side and the terrestrial surface is of the second order, for it $= ds. \sin. i$, or simply $i. ds.$ and since this geodesic line is equal to the right line, it necessarily follows, that it is the shortest which can be traced on the earth between any two points, it therefore measures the itinerary distances of places; its curvature likewise exists in a plane at right angles to the horizon, as is evident from the manner in which it has been traced. It is evident from what precedes that the difference between the length of this line and that of the corresponding arc of the terrestrial meridian may be neglected. Another property of the geodesic line is, that the sines of the angles

made by the perpendicular with the respective meridians are inversely as the ordinates of the point of concourse.

It is clear that when the earth is a solid of revolution, all the normals to the surface of this solid meet the axis of rotation, consequently those which pass through the points of the generating curve are necessarily in the plane of this curve, and $\therefore$ in that of the celestial meridian.

(*i*) Calling a b the equatorial and polar semidiameters, ρ ρ' the corresponding radii, t t' the two tangents, &c. c the arc of the evolute, then $a = \rho + t$, $b = \rho' - t'$, $\therefore a - b = \rho - \rho' + t + t'$.

(*k*) In determining the position of places in a region of considerable extent, it is necessary first to traverse it with a meridian line, from one extremity to the other, on this a certain number of points are selected, through which perpendiculars to the meridian are drawn. The meridian and its perpendiculars in this manner constitute a system of cervilinear coordinates, to which the different points of the earth's surface may be transferred. The great advantage of this method is, that when the extent of the region is not very considerable, these perpendiculars may be considered as great circles, and distances measured on them are the shortest between two given points.

(*l*) The method indicated in the notes to page 212 is perhaps the best and simplest of all, however it cannot be always applied; in that case, other methods have been devised, all of which may be reduced to the solution of certain cases of obtuse angled spherical triangles. Such as from having two altitudes of the sun, and the time between, or from observing the zenith distances of a heavenly body when near the zenith, the latitude is determined; the method which employs two altitudes of the sun has the advantage of enabling us continually to approximate to the true value.

(*m*) In fact the longitude and latitude only give the projection of a place on the earth's surface, but do not define its position in space; in order to determine this we must

know the elevation of the place about the level of the sea. A determination of the heights of the most remarkable places in Europe would, combined with a knowledge of their longitude and latitude, be a more complete way of levelling than by trigonometrical operations, and would perfectly point out the directions of chains of mountains, and also the falls of rivers, &c., and thus give a most accurate notion of the form of the earth. As illustrative of the utility of these kind of observations, it may be remarked that a comparison of the heights of the barometer in the Euxine and Caspian Seas, evince that the level of the latter is considerably lower than the former.

(*n*) The repeating circle is an invaluable instrument to the practical astronomer, it supplies the place of a mural quadrant, and also of a transit instrument; besides it is capable of almost indefinite exactness, and from the smallness of its size it may easily be transported from one place to another.

(*o*) In general the retardation of time is proportional to the angle contained between the meridians of the two places, hence appears the reason of what has been already adverted to, namely, that if while one observer be fixed, another proceeds round the earth, he will on his arrival at the place from whence he set out, have either gained or lost a day, according as he went, eastward or westward.

(*p*) The chronometers now in use, being furnished with compensators, which secure them from the effects arising from changes of temperature, and also from the inevitable effects of the agitation which they experience during a long voyage, give the time with extreme accuracy.

The *true* time H at the place of observation is easily obtained when the latitude of the place or vessel, Z the zenith distance or altitude of the star, and d, its declination are given, for it is easy to show that

$$\text{Sin.}\ \frac{H}{2} = \sqrt{\frac{\text{Sin.}\ \frac{(Z+P-D.)}{2}\ \text{Sin.}\ \frac{(Z+D-P.)}{2}}{\text{Sin.}\ P.\ \text{Sin.}\ D.}}$$

See Notes, p 304, 292. But as the chronometer indicates *mean* time we must apply the equation of time in order to obtain the mean time at the place of observation. This method assumes that the time indicated by the chronometer is exact, which is not the case; however its rate of going and small inequalities may be ascertained by comparing it with the time pointed out by observing the altitudes of the sun or stars as often as possible.

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the respective latitudes, and D the difference of longitudes, cos. a the mutual distance $=$ cos. λ. cos. λ'. sin. D$+$sin. λ. sin. λ'. This is on the hypothesis that the earth is $q.\ p.$ circular; if it be supposed to be an ellipsoid of revolution, the direction of verticals from the two places do not meet in the same point of the axis, and ∵ do not make a solid angle; in that case we deduce the angles which rad. from centre of ellipsoid to the two places make with the axis, and the inclinations of the planes of these angles to each other is also given, hence the angle which the rad. vectors make with each other may be determined, and hence the mutual distance of the places, the distance of each place from the centre being known.

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In fine the great objection against the French system is, that it depends upon an accurate measure of a quadrant of the meridian, at the same time that no such measure has hitherto been obtained, besides the meridians differ so widely among themselves that it is likely that no accurate mean length of the pendulum will ever be obtained.

The idea of verifying a standard by some other means than by a comparison with some actually existing standard, though suggested a great while ago, was never completely acted on, until the new system of weights and measures was introduced into France.

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(*c*) Let z represent the vertical height, m' the density, g' the gravity, p the pressure or elastic force of the air, x the temperature, we have $adp = m'g'dz$, $\therefore$ $-\frac{dp'}{p} \backsim -dz'$, $\because$ log. $p \backsim$ as z, and if z be taken in arithmetical progression, the Naperian logarithm of $\frac{1}{p}$ is in arithmetical progression, and $\because \frac{1}{p}$ is in geometric progression, and as the densities are as the compressing forces, *i. e*, as the heights of the mercury in the barometer, in the same circumstances these heights will decrease in geometrical progression (*a* expresses the ratio of the elastic force to the density, when the temperature is zero, and is evidently the same for the same elastic fluid, but is different for each) and $\because$ if h, h' are the columns of the mercury at the surface, and at any elevation z from the surface, K representing the constant coefficient to be determined by experiment, we have $z = K.(\text{log. } h - \text{log. } h')$ $= K. \text{log. } \frac{h}{h'}$, hence K will be had if z is determined

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(h) From these weights the ratio of the specific gravity, and ∴ the constant coefficient may be deduced, which ought to agree with the coefficient deduced *a priori* from a comparison of the same height as furnished by barometrical and trigonometrical observations, but these disagree; and as this disagreement cannot be accounted for by introducing the consideration of humidity, the variation of the force of gravity as we ascend from the earth must be taken into account; this diminution of the force of gravity will be taken into account, if in the equation $adp = m'.g'.dz$, we substitute $\frac{gr^2}{(r+z)^2}$ for g', then we have $\frac{dp}{p} = -\frac{gr^2}{a.(1+ax)} \cdot \frac{dz}{(r+z)^2}$, which gives by integrating, log. $p = \frac{K.gr^2}{a.(1+ax)} \times \frac{1}{r+z} + C$, x is supposed to be constant, and ∴ if π be the value of p, when $z = 0$; log. $\frac{\pi}{p} = \frac{K.gr}{a.(1+ax)} \times \frac{z}{r+z}$, hence when z is known, and the heights in the barometer observed, we can determine

k; by means of this equation, combined with the value of p, given in Note (c), we can determine the laws of density and elastic force of the air for a given state of equilibrium of the atmosphere. Now in order to apply this formula to the mensuration of heights we have $\pi = mgh$, $p = mg'h'\left(1 + \frac{T-T'}{5412}\right)$, T, T', being the temperatures of the mercury at the two stations; in order to abbreviate, let h' represent the height of the barometer at the second place of observation multiplied into $1 + \frac{T-T'}{5412}$, then we have $\frac{\pi}{p} = \frac{h}{h'} \cdot \frac{(r+z)^2}{r^2}$; $\therefore \log. \frac{\pi}{p} = \log. \frac{h}{h'} + 2 \log. \left(1 + \frac{z}{r}\right)$; let t, t' be the respective temperatures of the air, which differ from T, T', as a given difference of temperature, is not so rapidly communicated to the mercury as to the external air, $x = \frac{t+t'}{2}$, $a = ,004 = \frac{1}{250}$, $\therefore ax = 2 \cdot \frac{(t+t')}{1000}$; hence substituting these values we get $z = \frac{a}{Kg} \cdot \left(1 + \frac{2.(t+t')}{1000}\right) \cdot \left(\log. \frac{h}{h'} + 2 \log. \left(1 + \frac{z}{r}\right)\right)$; $\frac{a}{Kg}$ is the coefficient 18336, mentioned in the text, it is obtained by an equation of condition which is given from knowing z, h, h', t, t', and r the radius of the earth; this value is for the latitude 45, for any other $\frac{a}{Kg} = 18336$ (1,002837 cos. 2ψ). In the determination of z, as $\frac{z}{r}$ occurs in the second member, where it is an extremely small fraction, we 1st compute z on the supposition that this fraction is wanting, we then substitute the value of $\frac{z}{r}$ determined in this supposition, and as it is extremely small, the result differs inconsiderably from the

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(*f*) The intervals between which it has been ascertained from experiment that this = dilatation obtains, is from zero to 100° of the centigrade thermometer, and it is even true for those aeriform substances which are produced by vaporization, provided that they are not charged with any liquidity, hence 00375 being represented by a, and the increase of temperature by x, we have $p = am'.(1+ax)$.

(*g*) The aqueous vapours are necessarily less dense than the air in which they float, and from their being mixed in the air, it is enabled to sustain with a less density a column of mercury of the same height, ∴ this vapour weighs less than dry air, perfectly free from humidity, of the same elastic force. *See* Note (*v*) of this Chapter.

(*h*) From these weights the ratio of the specific gravity, and ∴ the constant coefficient may be deduced, which ought to agree with the coefficient deduced *a priori* from a comparison of the same height as furnished by barometrical and trigonometrical observations, but these disagree; and as this disagreement cannot be accounted for by introducing the consideration of humidity, the variation of the force of gravity as we ascend from the earth must be taken into account; this diminution of the force of gravity will be taken into account, if in the equation $adp = m'.g'.dz$, we substitute $\frac{gr^2}{(r+z)^2}$ for g', then we have $\frac{dp}{p} = -\frac{gr^2}{a.(1+ax)}\cdot\frac{dz}{(r+z)^2}$, which gives by integrating, log. $p = \frac{K.gr^2}{a.(1+ax)} \times \frac{1}{r+z} + \text{C}$, x is supposed to be constant, and ∴ if π be the value of p, when $z = 0$; log. $\frac{\pi}{p} = \frac{K.gr}{a.(1+ax)} \times \frac{z}{r+z}$, hence when z is known, and the heights in the barometer observed, we can determine

K; by means of this equation, combined with the value of p, given in Note (c), we can determine the laws of density and elastic force of the air for a given state of equilibrium of the atmosphere. Now in order to apply this formula to the mensuration of heights we have $\pi = mgh$, $p = mg'h'\left(1 + \frac{T-T'}{5412}\right)$, T, T', being the temperatures of the mercury at the two stations; in order to abbreviate, let h' represent the height of the barometer at the second place of observation multiplied into $1 + \frac{T-T'}{5412}$, then we have $\frac{\pi}{p} = \frac{h}{h'} \cdot \frac{(r+z)^2}{r^2}$; $\therefore$ log. $\frac{\pi}{p} =$ log. $\frac{h}{h'} + 2$ log. $\left(1 + \frac{z}{r}\right)$; let t, t' be the respective temperatures of the air, which differ from T, T', as a given difference of temperature, is not so rapidly communicated to the mercury as to the external air, $x = \frac{t+t'}{2}$, $a = ,004 = \frac{1}{250}$, $\therefore ax = 2 \cdot \frac{(t+t')}{1000}$; hence substituting these values we get $z = \frac{a}{Kg} \cdot \left(1 + \frac{2.(t+t')}{1000}\right) \cdot \left(\text{log.}\ \frac{h}{h'} + 2 \text{ log.} \left(1 + \frac{z}{r}\right)\right.$; $\frac{a}{Kg}$ is the coefficient 18336, mentioned in the text, it is obtained by an equation of condition which is given from knowing z, h, h', t, t', and r the radius of the earth; this value is for the latitude 45, for any other $\frac{a}{Kg} = 18336\ (1,002837\ \cos.\ 2\psi)$. In the determination of z, as $\frac{z}{r}$ occurs in the second member, where it is an extremely small fraction, we 1st compute z on the supposition that this fraction is wanting, we then substitute the value of $\frac{z}{r}$ determined in this supposition, and as it is extremely small, the result differs inconsiderably from the

truth. Besides the corrections mentioned in the text, when extreme accuracy is required we must take into account the convexity of the mercury in the upper part of the tube, and also the effect of capillary attraction. With respect to the cause of the variation of the length in the barometric column, various theories have been suggested, none however completely satisfactory. In the Notes to Chapter X. Vol. II. we shall enter into some details respecting the periods, &c. of these variations.

By very precise experiments made with the hygrometer, it has been ascertained that the power of the air to retain moisture is doubled at every increase of temperature of the centrigrade thermometer by 15 degrees, or in other words, while the temperature increases in an arithmetical progression, the quantity of moisture which the air is capable of holding in solution increases in a geometrical ratio; these indications of the hygrometer do not point out the absolute degrees, but only its relative dryness with respect to the ball of the hygrometer.

It has been computed, that if the atmosphere would pass from its point of saturation in dampness, to a state in which the air would be completely destitute of humidity, the whole quantity of water discharged would not constitute a sheet of water five inches in depth.

(*i*) The natural colour of the air is blue; but in order to be apparent, the depth of the air should be considerable. This is the reason why the colours of very distant objects are always tinged with the blue of the intermediate atmosphere. In fact, as the particles which compose the air are extremely small, and at a distance from each other, they could not be perceived unless they were united in a mass; conformably to this, it is found that according as we ascend in the atmosphere, the blue colour becomes less brilliant, for the brightness diminishes with the density af the air which reflects it, so that on the summit of a high mountain, or to an aeronaut, the sky appears black.

As no coloured substance discloses its inherent colour, but by separating the rays of light, in order that its real colour should be exhibited the particles of light must penetrate the atmosphere, and after undergoing some change be again emitted. In the atmosphere, besides the internal dispersion of the blue rays, the white light is reflected in various quantities without any change, as is evident from the phenomena of polarization. And as the white light, in its transit through the air, continually loses more and more of the blue rays, it must, according as it advances, assume the complimentary colours of the spectrum, and ∵ become successively yellow, orange, red and crimson.

(*k*) It is the reflective power of the atmosphere, which makes objects to appear uniformly enlightened in *every* direction; if it had not this power, the bright sides of objects would be only visible, and their shadows would be, in all probability, insensible. The evening twilight is longer than the morning, because the atmosphere is then more dilated by heat.

The last ray which comes to the spectators eye touches the earth when it is first emitted from the sun; and secondly, when it reaches the spectator after being reflected at the extreme verge of the atmosphere.

In this method allowance should be made for the inflection of the ray, or for its deviation from a rectilinear course by the action of the continually denser strata. For the greatest height of the atmosphere at the equator, *see* Vol. II. Chapter XIII.

If the density of the air decreased in geometric progression at fifteen miles elevation, the height of the barometer would be only one inch; ∵ the greatest part of the atmosphere is always within fifteen, or at farthest twenty miles of the earth, and ∵ though from the refraction of the sun's light, and from the duration of twilight, it has been inferred that the height is from forty to forty-five

miles; Wollaston thinks that it is limited to the former height; in fact, the force of gravity on a single particle is then equal to the resistance which arises from the repelling force of the particles of air; hence he likewise infers, that there is a limit to the divisibility of matter.—*See* Philosophical Transactions, 1822.

On the contrary, a stratum of air at five and a half miles depth from the surface, would have such a density that it would never rise to the surface; $\because$ as the mean depth of the sea, as given by the theory of the tides, see Vol. II. Chap. XII., is twice that quantity, the conjecture of some philosophers may be true, that the bed of the ocean rolls on this subaqueous air, which, though it never rises to the surface, may support the combustion which we know goes on below the surface of the earth.

It is easy to compute the duration of twilight, when the latitude and declination are known,; for as it appears from repeated observations, that it lasts until the sun is 18° below the horizon, if h', h, represent the hour angles at the termination and beginning of twilight, we have $\cos. h = -\tan. l. \tan. \delta$, $\cos. h' = -\frac{\sin. 18^\circ}{\cos. l. \cos. \delta} - \tan. l. \tan. \delta$, $\because \cos. \frac{1}{2}(h'-h) = -\frac{\sin. 18^\circ}{2 \sin. \frac{1}{2}(h'+h). \cos. h. \tan. l}$, $\because$ it is shortest when l and $\delta = 0$, as the greatest depression of the sun $= 90 - (l+\delta)$, if this quantity is less than 18, or $72 \angle l + \delta$, twilight will last all night, or rather the morning twilight will immediately succeed the evening. Cos. h' is always > 90 until l and δ are of opposite affections, and $\sin. l. \sin. > \sin. 18^\circ$, or $\sin. l > \frac{\sin. 18^\circ}{\sin. 23^\circ, 28'}$, $\because l > 50^\circ, 54$; hence all parts of the earth, of which the latitude exceeds 51°, have the days longer than the nights in consequence of this power of the air to reflect light, and at the poles it lasts until the sun is 18° at the other side the equator, so that the two twilights, be-

fore and after the commencement of summer, last fifteen weeks; if cos. $(h'-h)=-1$, twilight lasts all night; in this case, sin. $18=\cos.(l+\delta)$, $\because\ \delta=72-l$, hence, the part of the year during which twilights can last all night increases with l, and its least value is 48°, 32′. To determine the day in which, in a given place, the duration of twilight may be given quantity. Let $h'-h=\gamma$ then we have $\cos. h=-\frac{\sin. l.\sin.\delta}{\cos. l.\cos.\delta}$, and $\cos.(h+\gamma)$

$$=\cos. h'=-\frac{\sin. 18^\circ+\sin. l.\sin.\delta}{\cos. l.\cos.\delta},\ i.e.\ \cos. h.\cos.\gamma.$$

$$-\sin. h.\sin.\gamma=\cos. h-\frac{\sin. 18^\circ}{\cos. l.\cos.\delta},\ \because\ \sin. h=$$

$$\frac{\sin. 18^\circ+\sin. l.\sin.\delta.(1-\cos.\gamma)}{\sin.\gamma.\cos. l.\cos.\delta}=\frac{\sqrt{\cos.^2 l-\sin.^2\delta}}{\cos. l.\cos.\delta};$$

$\because$ by squaring and concinnating we obtain $(\sin.^2\delta.(2\sin.^2 l.(1-\cos.\gamma)+\cos.^2 l.\sin.^2\gamma)+2\sin.\delta.\sin. 18^\circ\sin. l.(1-\cos.\gamma)+\sin.^2 18-\cos.^2 l.\sin.^2\gamma=0$; the solution of this equation gives two different values of δ, and as the sun has the same declination twice every year, there are four different days in which the duration of twilight is the same. To find the shortest twilight, we have by differentiating the preceding values of cos. h, and cos. $(h+\gamma)$ supposing γ and δ to vary, $dh=\frac{d\delta.\sin. l}{\cos. l.\cos.^2\delta.\sin. h}$, and

$$dh+d\gamma=\frac{d\delta.(\sin. l+\sin. 18^\circ.\sin.\delta)}{\cos. l.\cos.^2\delta.\sin.(h+\gamma)},$$ $\because$ as γ is supposed to be a minimum, $d\gamma=0$, $\because\ \frac{\sin.(h+\gamma)}{\sin. h}=$

$$\frac{\sin. l+\sin. 18^\circ.\sin.\delta}{\sin. l},\ \text{but}\ \sin. h=\frac{\sqrt{\cos.^2 l-\sin.^2\delta}}{\cos. l.\cos.\delta},$$

and $\sin.(h+\gamma)=$

$$\frac{\sqrt{\cos.^2 l-\sin.^2\delta-2\sin. 18^\circ.\sin. l.\sin.\delta-\sin.^2 18}}{\cos. l.\cos.\delta},$$

$$\therefore \frac{\sin.(h+\gamma)}{\sin.h} =$$

$$\frac{\sqrt{\cos.^2 l - \sin.^2\delta - 2\sin.18^\circ.\sin.l.\sin.\delta - \sin.^2 18}}{\cos.^2 l - \cos.^2\delta},$$

$\therefore$ equalling these two values of $\frac{\sin.(h+\gamma)}{\sin.h}$, squaring and dividing by $\sin.^2 18^\circ.\cos.^2\delta$ we obtain $\sin.^2\delta + \frac{2\sin.l.\sin.\delta}{\sin.18^\circ} + \sin.^2 l = 0$, $\therefore \sin.\delta = -\frac{\sin.l.(1 \mp \cos.18^\circ)}{\sin.18^\circ}$ $=$ either $-\sin.l.\tan.9^\circ$, or $-\sin.l.\cot.9^\circ$, $\therefore$ the shortest twilight occurs four times in the year, and always in winter time, for δ is negative; but as δ cannot exceed 23,28, in the second value of δ, if $\sin.l$ is $>r$ than $\tan.9^\circ.\sin$! 23,28 it is impossible, $\therefore$ this solution only obtains for latitudes less than $3^\circ,37'$, but the first is true for all latitudes for its maximum value, *i. e.* when $l = 90$, is $\sin.\delta = \tan.9^\circ$; this would appear therefore to determine δ for the shortest twilight under the pole; however this problem is not applicable to the pole, as we can have but one day, and consequently but one twilight under the pole during the entire year; in general that several twilights may occur successively, it is necessary that $180 - l + \delta > 108$, *i. e.* that $l < 72 + \delta$; $\therefore$ conformably to this condition, it results from the first value of δ, that $\sin.\delta \angle$ than $\tan.9^\circ.\sin.(72+\delta)$ or $\tan.\delta \angle \frac{\tan.9.\cos.18}{1-\tan.9.\sin.18}$, or $\tan\delta \angle \tan.9$; $\therefore l$ is less than $72+9$, or 81 $\therefore l+\delta \angle 90^\circ$; this shews that the firs root is not applicable to all the earth, for all places whose latitude is $>$ than 80°, the sun does not set for the day of shortest twilight; it is evident that if $l = 0, \delta = 0$, $\therefore$ the shortest twilight at the equator is when the sun is in the equator. To find the duration of the shortest twilight, let the angles formed by the vertical and circle of declination at the sun set and at the end of twilight $=$ s and S respectively, then we have $\cos.s = \frac{\sin.l}{\cos.\delta}$, $\cos.S =$

$\frac{\sin. l + \sin. 18. \sin. \delta}{\cos. 18. \cos. \delta}$, substituting for sin. δ its value — tan. 9. sin. l, we have $\cos. S = \frac{\sin. l.(1 - 2. \sin.^2 9^\circ)}{\cos. 18. \cos. \delta} =$ $\frac{\sin. l. \cos. 18^\circ}{\cos. \delta. \cos. 18} = \cos. s$, $\because s = S$, $\because$ in the vertical passing through the sun, if an arc $= 18^\circ$ be taken, it is easy to prove that the zenith distance is equal the arc of a great circle, formed by lines from the pole to the extremity of this arc, and that the angle between them $= \gamma$, $\because$ in this isoceles triangle we have $\cos. \gamma = \frac{\cos. 18 - \sin.^2 l}{\cos.^2 l}$

$\therefore 1 - \cos. \gamma = 2 \sin.^2 \tfrac{1}{2}\gamma = \frac{1 - \cos. 18}{\cos.^2 l} = \frac{2 \sin.^2 9}{\cos.^2 l}$

$\because \sin. \tfrac{1}{2}\gamma = \frac{\sin. 9}{\cos. l}$.

(*o*) A ray of light is made to pass through a prism, out of which the air is supposed to be completely excluded, and if the sides of the prism be perfectly parallel, the deviation which the ray experiences must arise from the refraction of the external air; and from knowing this deviation, and also the refracting angle of the prism, the ratio of the sine of incidence to the sine of refraction can be determined for gases or liquids.

There is however this difference, that in case of gaseous substances the refracting angle of the prism may be considerably greater than for liquids; in the latter it cannot exceed a certain limit, which is thus determined, sine of half the angle of prism is to radius as sine of incidence to sine of refraction from the liquid into air.

It is easy to shew that for any ray refracted by the prism, the sine of the deviation of the ray is to the sine of refracting angle of the prism, as sine of incidence is to the difference between the sine of incidence and the sine of refraction from the prism into air. It is

in this manner that the ratio of the sine of incidence to the sine of refraction is determined in page 145 of the text.

As it is nearly impossible to procure a perfect vacuum, the height of the mercury in the gage must be observed, and account made of it in the calculus. If we wished to obtain the refraction of the air at different densities it would be only necessary to note the height of the mercury in the gage at the respective densities; or if the refractions of other gases were required, we should exhaust the prism as far as possible, and then after noting the height of the mercury in the gage, introduce the gas. Caustic potash is generally introduced to absorb the aqueous vapours which exist in the air, when its density is so reduced; on the contrary, if the refractions of aqueous vapours were required, we should charge the atmosphere with them, by means of vessels of water and of moistened towels. See *Biot's Physique*, tom. 3.

(*p*) The diversity of colours arises from the particular disposition of bodies to reflect some rays rather than others. When this disposition is such that the body reflects every kind of ray in the mixed state in which it receives them, that body appears white; ∵ white is not a colour, but rather the assemblage of all colours; if a body has a disposition to reflect one sort of rays more than others, by absorbing all the others, it will appear of the colour belonging to that species of rays. As different bodies are fitted to reflect different kinds of rays, they must appear of different colours; when a body absorbs all the light which reaches it, it appears black, as it transmits so few reflected rays that it is scarcely perceivable.

(*q*) The density of the atmospherical strata decrease in arithmetrical progression, when the temperature diminishes in arithmetrical progression; for the density m being equal to Q the quantity of matter divided by the volume, if 1 represent the volume previously to x the in-

crease of temperature, we have $m = \frac{Q}{1+ax}$, (a representing 0,375) $= Q.(1-ax)$ nearly; $\because$ when the increments of the temperature are given, the densities decrease by an arithmetical progression.

There are two causes of the decrease of heat, according as we ascend in the atmosphere, namely, our receding from the earth, the principal source of heat, and also from the circumstance of the air being less compressed, which makes its absorbing power greater. But though the heat thus decreases in a less ratio than the distances increase, still the rate of decrease is nearly uniform when the height is inconsiderable.

When the altitude exceeds eleven degrees the inclination of a ray of light to the atmospheric strata is less oblique, consequently the curvature of the portion of the trajectory to be described by the star is less, and according as the altitude increases, it approaches, more and more to the rectilinear direction; now if the strata of the atmosphere were parallel to each other, and to the earth, considered as a plane, the refraction would be what would take place if the ray passed from a vacuo into air of the same density as that at the earth's surface; the error, $\therefore$ arises from the earth being supposed to be a plane, when it is in point of fact spheroidical, which shape is communicated to the atmospherical strata. In the former case, the refraction would depend on the total increase of density of the atmosphere, *i. e.* on the pressure and temperature which are indicated by the barometer and thermometer.

It may likewise be observed here, that when the elevation is greater than eleven degrees, the differential equation of the trajectory described by the ray of light, namely $dr = dv\sqrt{Q}$ (where r is the radius from the centre to any point of the trajectory, and v the angle between r and a vertical at this point, Q a function of r depending on the law

of the decrease of densities) may be expressed in a very convergent series; but when the trajectory is horizontal, dr and ∴ $\sqrt{Q}=0$; ∴ if it is near to a horizontal state, Q is inconsiderable, ∴ $\sqrt{Q}$ cannot be developed in a convergent series, because the several terms which compose it have a finite ∸ to each other; but when the point is at a considerable distance from its *minimum* state, some of the terms composing Q are considerably greater than others, ∴ the expansion of $\sqrt{Q}$ into a series is possible, and ∴ the equation of the trajectory may be obtained by approximation. —*See* Mechanique Celeste, tom. 4, livre 10.

(s) If $n:1$ be the ratio of sin. I to sin. R from a vacuum into air, we have, if ir be the angles of incidence and refraction, z the zenith distance, a the radius of the earth, and h the height of the homogeneous atmosphere, $a+l : a ::$ sin. $z :$ sin r; $1 : m :$ sin. $r :$ sin. i, ∴ sin. $i = \frac{m.a.\ \text{sin.}\ z}{a+l} = m.$ sin. $z.\left(1-\frac{l}{a}\right)$; sin. $r =$ sin. $z\left(1-\frac{l}{a}\right)$; $i = r + R$; ∴ sin. $(r+R) =$ sin. i; and sin. r + cos. r. sin. $R = m$. sin. r, ∴ $(m-1)$. tan. $r =$ sin. R, or R; hence substituting for sin. r, and sin. i, and also for cos. $r = \sqrt{1-\text{sin.}^2 r}$, $=$

$$\sqrt{1-\text{sin.}^2 z\left(1-\frac{l}{a}\right)^2} = \sqrt{\text{cos.}^2 z + \frac{2l\text{sin.}^2 z}{a}} = \text{cos. } z.$$

$$\left(1+\frac{l}{a}.\ \text{tan.}^2 z,\right) \therefore R = \frac{\text{sin. } i - \text{sin.} r}{\text{sin. } 1''.\ \text{cos. } R}$$

$$= \frac{(m-1).\ \text{sin. } z.\left(1-\frac{l}{a}\right)}{\text{sin. } 1''.\ \text{cos. } z.\left(1+\frac{l}{a}.\text{tan.}^2 z\right)} = \frac{(m-1).\ \text{tan. } z}{\text{sin. } 1''}$$

$-\frac{(m-1).l\ \text{tan.}^2 z}{a.\ \text{sin. } 1''.\ \text{cos.}^2 z}$. If $z = 80°$, $l = 5$, $a = 4000$ miles, the second term will not exceed $10''$; this is what arises from the spherical shape of the earth; if a was infinite,

i. e, if the earth was a plane, it would vanish; ∵ as far as 80° of zenith distance, the error from the supposition that the density of the atmosphere is uniform, and the earth a plane, must be less than 10″. Now from the ratio of sin. i : sin. r from a vacuum into air, m the coefficient of the refraction may be determined (p. 369). This coefficient is as the refractive force of the air, *i. e*, as its density, or as $\frac{Q}{M}$; ∵ to reduce the coefficient to a given temperature and pressure, it must be divided by $1+a\tau$. (*see* page 370), and then multiplied by the direct ratio of the pressures, ∵ the true coefficient $= \frac{p}{,0,76(1+at)}$; but if these quantities are determined for the latitude of 45, they should be multiplied by cos. 2ψ for any other latitude ψ.—(*See* p. 341.)

(*t*) It may be doubted whether the analysis given in the text is complete, for a recomposition of these materials will not give air of the same nature as the atmosphere, ∵ some of the elements or constituents must have escaped during the decomposition, which is indeed probable, as the air is charged with emanations from the various substances with which it comes in contact; we are certain, as was before observed, that the quantity of aqueous vapour is not always the same; it appears from this that, if its chief constituents are always in the same ÷n, the purity or insalubrity of the air must depend on somethiug besides this proportion. It is conjectured with some degree of probability, that the higher regions consist ofi nflammable materials, which is the cause of those appearances which it frequently exhibits, namely, of shooting stars, fire-balls, and luminous arches; these materials arise from the numerous emanations from volcanoes, &c. &c., and as hydrogen is lighter than common air, and has very little affinity for its constituents, it ascends upwards from its greater levity, and from the extent and celerity of these

phenomena they must necessarily take place in the most elevated regions of the atmosphere: this conjecture is confirmed by astronomical refractions, for the refraction in these elevated regions is greater than what computation assigns to them, but on the supposition that hydrogen gas is one of their chief constituents, this discrepancy disappears, for the refraction of this gas is greater than that of other substances in proportion to its density, while oxygen gas is the least refractive of the gases.

Chemists are not agreed as to the manner in which the constituents of the atmosphere exist in it; some suppose that they are chemically united, chiefly from the uniform manner in which they are always combined, and because they are not arranged according to their respective specific gravities; others think that the particles of the gases which compose the atmosphere neither attract nor repel one another, and that the weight on any one particle of the atmosphere arises solely from particles of its own kind. —*See* Manchester Memoirs, p. 538.

(u) It is easy to find the stratum of air, of which the density is such as is described in the text, for let c^3 be the capacity of the balloon, y the specific gravity of the stratum of air in which the balloon floats in equilibrio, since a cubic foot of water weighs 62,48lbs, $c^3. (62.48)y$ is the weight of the displaced air, and the whole weight is $w + (62.48).\ c^3.\ \frac{y}{13}$, when these quantities are $=$, we can determine y, and $\because$ the density of the stratum, and consequently the height, from knowing the density of the air and height of the mercury at the earth's surface.

(Note. w is the weight with which the balloon is loaded, and the hydrogen gas which is generally used is only six times lighter than common air.)

Besides the circumstances mentioned in page 151, it was ascertained, as mentioned above, that the elasticity of the upper regions of the atmosphere was greater than near the earth's surface, also the diminution in the tempe-

rature was less than what was experienced in corresponding heights on the earth's surface, and the indications of the hygronometer shewed that the atmosphere became dryer according as we ascended; but indeed this might arise from the increased attraction of the air for moisture in consequence of its less density.

(*v*) Knowing the refractive power of water, from note page 372, we can determine it for water reduced to vapour of the same density as the air, for these refractions are ∻l totheir densities; now the density of this vapour would give its refraction greater than that of air; but as the density of the vapours which float in the air are less than that of air, the refraction of the vapour must be diminished, and by nearly the quantity by which it was greater than that of air. Biot made his direct experiments on the refraction of air saturated with humidity, and at high temperatures. Note, there are some exceptions to the position that the refractions are ∻l to the densities, for it is not true for the class of inflammable substances.

Suppose for instance, as stated in page 153, that a wind blew for a long time in the same direction, the curvature of the inferior strata would be necessarily affected by it, and ∵ the refractions computed from it would be very unequal. The temperature may produce equally anomalous effects, as for instance, if from the greater heat of the surface of the earth, the density of the lower strata was less than of those more elevated, as is the case in the phenomena observed frequently in Egypt, which are called *mirages*.

The effects of diurnal parallax and refraction are very different, and may easily be distinguished one from the other; as refraction elevates, and parallax depresses;—the first increases and the second diminishes the duration of the visibility of the stars above the horizon. Each is greatest at the horizon, but as the refraction varies nearly

as the tangent of the zenith distance, near to the horison it varies very irregularly and with great rapidity, and near the zenith slowly and regularly; on the contrary, near the zenith the variation of the parallax is quickest, and slowest near to the horizon; as the refraction of the sun is greater than his parallax, we enjoy his light longer than if these effects did not exist, on the contrary, the parallax of the moon being greater than the refraction, we enjoy the light of the moon for a shorter time than without these effects. At the horizon refraction diminishes the vertical and horizontal diameters of the sun and moon; the diminution of the latter is insensible, but that of the former is more than 4′; both are nearly insensible when the altitudes are more than ten degrees. Parallax increases both diameters, at the horizon however the quantity is insensible; on the contrary, at the zenith the vertical diameter of the moon is increased a sixtieth part. From the horizontal refraction of the sun being greater than the corresponding diameter, we see the entire disk when it is beneath the horizon, and a spectator at the poles will see the sun two days sooner than if it did not exist.

(x) Let X, X', X'', &c. represent the light in the 1st, 2d, 3d, &c. strata of air, as the same quantity, namely its $\frac{1}{t}$ part, is supposed to be lost in each of those = strata, we have $X-\frac{X}{t}=X'$, $X'-\frac{X'}{t}=X''$, $\therefore \frac{(t-1)}{t}.\ X=X'$, $\frac{(t-1)^2}{t}.\ X=X''$, &c.; hence the logarithms of the intensity of light are $\div$l to the thickness of the stratum; in fact, ε denoting the intensity of light at any stratum we have, $d\varepsilon=-A\varepsilon.m.\sqrt{dr^2+r^2.dv^2}$, where m denotes the density of the stratum r its radius, and v the zenith distance, $\therefore \frac{d\varepsilon}{\varepsilon}=-A\rho.\sqrt{dr^2+r^2dv^2}$; now rdv is of the

form $\frac{1}{\cos. v}$; when the altitude of the star is greater than 12°; $\frac{d\varepsilon}{\varepsilon} = -\frac{A\rho.dr}{\cos. v}$, and log. $\varepsilon = -\int \frac{A\rho.dr}{\cos. v}$. Let E be the value of ε in the zenith where cos. $v = 1$; then we have log. $\varepsilon = \frac{\log. E}{\cos. v}$; log. E $\backsim (\rho). l$, *i. e.* it is :: to the height of the barometer.

From some observations, founded on the preceding analysis, it was inferred, that at the altitude of 25° when the sky is most serene, the sun loses $\frac{1}{2}$ of its light, and at an elevation of 15° it loses $\frac{1}{3}$ of its light.

The continual agitation of the atmosphere produces momentary condensations and dilatations in the particles composing it, which causes the direction of the luminous rays to vary continually from the diversity of refractions which they occasion.—*See* Notes to page 362.

(*a*) In note (*c*) to page 317, it was stated that the height of the shadow was $= \frac{r}{\sin. (s-p)}$; but if the effect of refraction be taken into account, this expression should be $\frac{r}{\sin. (s+2R-p)}$; in like manner, the semidiameter of the section of the shadow $= p + P - s - 2R$; in the first expression, if s denote the distance of the centre of the sun from any point in its disk, it will determine the distance at which this point commences to be seen; if $s = 0$, we have the distance at which the centre of the sun becomes visible by the refraction at the earth's surface, or if s becomes negative, we have the distance at which points of the disk at the other side of the centre become visible; in like manner, by determining the value of s from the equation $p + P - s - 2R = 0$, we could determine the quantity of the sun's disk visible by refraction to a spectator at the moon, for any given distance from

the earth by computing P for this distance, and then determining s from the equation $P + p - s - 2R = 0$; from a computation made under the most unfavourable circumstances, it might be shewn that $\frac{3}{4}$ of the solar disk is visible by means of the earth's atmosphere.

Another effect of refraction was, that in consequence of it, the sun and moon were both so elevated in a total eclipse, as to be both visible at the same time.

BOOK THE SECOND.

CHAPTER I.

The arguments for the earth's rotation, which are detailed in this and the third chapter, may be reduced to the five following:—1st, the internal probability; 2d, the impossibility of the contrary; 3d, the analogy of the other planets; 4th, the compression of the earth, and the diminished lengths of isochronous pendulums as we approach the equator (which may be termed the physical proofs of this motion); 5th, the deviation of falling bodies to the east of the tower from which they are let fall. As shewing the far greater probability of the earth's rotation than that of the celestial bodies in a contrary direction, let us investigate the relative velocities of the earth and fixed star in the two hypotheses; the distance of the nearest fixed star is at least 200,000 radii of the earth's orbit (*see* Notes, page 337), its circumference, which is at least six times greater, is described in twenty-four hours; hence, it is easy to shew that its velocity is at least 6570 times greater than that of light; ∵ the star describes more than 270 milions of leagues, or more than twice the diameter of the earth's orbit in a second; and this velocity must be still greater, for the more distant stars, such as those which compose the milky way; on the con-

rary, supposing the earth to revolve, a point on the equator describes 5400 leagues in 24 hours, or in one second the sixtieth part of a league, which is a velocity a little greater than that of sound, and at least 4600 millions of times less than the preceding. Besides the motion to which, on the hypothesis of the earth's immobility, all celestial bodies must be subjected in order to explain the precession of the equinoxes, they must be in like manner subjected to another, in order to account for the nutation. Likewise, as all actions are accompanied with a contrary reaction, if the earth exerts a force to retain the celestial bodies in their diurnal paths, an = and contrary force must be exerted by them on the earth. And as the circles described by the stars are not concentric, but rather have their centres all existing in the axis of the earth, the central force should be different for each body; and as they all revolve in the same time, the force, whatever it is, should be greater for the more remote objects, contrary to what is observed in other cases of nature.

As an inhabitant of Jupiter would suppose the heavens to revolve in the time of Jupiter's rotation, so likewise an inhabitant of Saturn would come to the same conclusion for his planet, but one is inconsistent with the other. It is evident from the measurement of degrees, which was explained in the XIV. Chapter, that the earth is flattened at the poles; for a greater space must be traversed in the direction of the meridian near the poles than at the equator, in order to have the same inclination of two plummets.

If the earth be considered an ellipsoid, it is easy to prove that the attraction, or weight of a body, increases as we proceed from the equator to the poles, proportionally to the square of the sine of the latitudes (*see* Vol. II. Chapter VIII.); and if the earth revolves on its lesser axis, the centrifugal force, which is always perpendicular to this

axis, makes an angle which is continually more oblique, with the direction of gravity; and it is easy to shew that the part of this force which is efficacious varies very nearly as the square of the cosine of latitude; $\because$ the difference between the centrifugal force in equator and any parallel is $\propto$ to the square of the sine of the latitude; $\because$ in consequence of those two causes, the increase of weight from the equator to the poles must be $\propto$ to the square of the sine of the latitude; and the acceleration of falling bodies must increase in the same proportion, which is confirmed by experiments made with pendulums.—*See* Notes to Chapter II. Book III.

CHAPTER II.

(*a*) If light was progressive and not instantaneous, the last ray which issues from the satellite, at the commencement of the eclipse, or the first which we see at the termination of an eclipse, should strike our eye sooner in opposition, and later in conjunction, than if the eclipse occurred when the planet was at its mean distance from us. If the earth was in repose, a spectator on its surface would see a star in the direction of a ray of light issuing from the star; but if the earth be in motion, it is clear that in order to see the star, his telescope must be inclined to the direction of the first ray of light. If the ray and spectator were in motion in the respective directions of the light coming from the star, and of the direction of the earth's motion, the sensation or impression on the eye will be the same, as if the spectator was supposed to be at rest, and there was impressed on the ray, besides its own motion, that with which the spectator is actuated in a contrary direction, he would then see the star in the direction of the diagonal of a parallelogram, of

which the two previously mentioned motions constituted the sides, and the angle which this makes with the primitive direction of the ray of light is the aberration; ∵ if $1 : \mu$ be the ratio of the velocity of the earth to that of light, π the angle of aberration, ϕ the angle of the earth's way, we have $\sin. \pi = \frac{\sin. \phi}{\mu}$, μ is determined by the eclipses of Jupiter's satellites, and consequently for reflected light; however we shall see hereafter in Vol. II. Chapter XII., that the value is precisely the same for the direct light of the stars. Light traverses the diameter of the earth's orbit in 16′, 26″, 4‴; in this time the earth describes an arch = 40″, 5, ∵ velocity of light is to that of the earth as the diameter of a circle to an arc of 40″, 5, or as 2 to that number which expresses 40″, 5, in parts of the radius, ∵ $\frac{1}{\mu} = \frac{\sin. 40'', 5}{2} = \sin. 20''.25$, and $\pi = 20'', 25. \sin. \phi$, ∵ it is a maximum when ϕ is 90 or 270. As the diameter of the earth is 23000 less than that of its orbit, a point on the equator describes in a day a circle whose radius $= 1$; and in 365,25 days it describes a circle 23000 times greater, ∵ as the velocities are directly as the spaces and inversely as the times, the velocity of the annual motion is $\frac{2300000}{36525}$, or 63 times greater than that of the diurnal motion, and the diurnal aberration at the equator and at its maximum is $\frac{20''}{63}$ at most, *i. e.* less than a third of a second; and for any parallel of latitude χ, the coefficient $\frac{20''}{63}$, must be multiplied by $\cos. \chi$.

(*c*) The aberration of a fixed star takes place in a plane which passes through the star and the tangent to the earth's orbit, and is always in the direction of those parts towards which the earth moves, ∵ if the angle of the earth's way be acute, the star will appear elevated.

In the quadratures of the stars with the sun, relatively

to the earth, the aberration is made entirely in the plane of a circle of latitude passing through the star, so that the longitude is not at all affected; in the first quadrature the apparent latitude is β—20″. sin. β, in the last quadrature it is $\beta+20''$. sin. β, and their difference is 40″. sin. β; in the syzygies on the contrary, the plane of the circle of aberration is at right angles to the plane of the circle of latitude, and ∵ the latitude is not at all affected, whereas the longitude is most affected in those cases; hence it appears that the phenomena of aberration do not arise from the annual parallax.—*See* Notes to page 234. If a plane be conceived to pass through the star parallel to the plane of the earth's orbit, and if a line be drawn from the star parallel to a tangent at the earth, which may be to the stars' distance as the velocity of the earth to that of light, the star will always appear at the extremity of such line, and it will appear to trace the curve described by the extremity of this line, but as this line is ∻l to the velocity of the earth, and ∵ to the perpendicular let fall from empty focus on a tangent to the earth's orbit, it will appear to describe a curve similar to that traced by the intersection of the perpendicular with tangent, which curve is known to be a circle, ∵ a star viewed directly, or in pole of ecliptic, will describe a circle; between the pole and plane of ecliptic it describes an ellipse; and when in plane of ecliptic it describes an arc of a circle; the true place of the star divides the diameter of the circle, as the diameter of earth's orbit is divided by the sun. As the axes majores of the ellipses which the stars appear to describe are the same for them all; the velocity of the light as it emanates from them must be the same.

If λ be the longitude of a star, β its latitude, $\odot$ the longitude of the sun, the aberration in longitude is $-\frac{a.\cos.(\odot-\lambda)}{\cos.\beta}$, and the aberration in latitude $= a$. sin. $(\odot-\lambda)$. sin. β; the aberration in right ascension $=$

$-\frac{b. \cos. (\rho - \lambda) - c. \cos. (\rho + \lambda)}{\cos. \delta}$, the aberration in declination $= \sin. \delta. b. \sin. (\rho - \lambda) - c. \sin. (\rho + \lambda) - 8''. \cos. \beta. \cos. \delta$ (*see* Cagnoli, 1829;) hence we see that the aberration in longitude for a given star is a maximum when $\odot - \lambda$ is 0, or 180°, in which case the aberration in latitude vanishes, $\therefore$ it cannot arise from the parallax of the annual orb. In general the longitudes increase if $\odot - \lambda$ is between 90° and 270, and diminish in the first and last quadrants; the latitudes, whether northern or southern, diminish or increase according as $\odot - \lambda$ is $\angle$ or $>$ than 180. The greatest difference between the latitudes of a star arising from aberration $= 2a \cdot \sin. \beta$, the greatest difference of longitude $= 2a. \sec. \beta$, this increases to infinity for stars situated near the pole of the ecliptic.

The coefficient of aberration might be determined, *a priori*, suppose the change of declination in a star existing in the solstitial colure produced by aberration, be observed; in this case $\sin. \rho = 1$, $\cos. \rho = 0$, $\odot = 0$ at the vernal and 180 at the autumnal equinox, $\therefore$ the aberration at the vernal equinox $= a. \sin. (\delta - \varepsilon)$, and at the autumnal, the aberration $= -a. \sin. (\delta - \varepsilon)$, $\therefore$ the entire difference $D = 2a. \sin. (\delta - \varepsilon)$, and $a = \frac{D}{2. \sin. (\delta - \varepsilon)}$.

With respect to the coefficient a, as the motion of the ray of light is accelerated by the action of the transparent bodies, namely, the atmosphere, the object glass of the telescope and humours of the eye, which it must traverse before it reaches the retina, it follows that the value of a is not the velocity of the ray when it enters our atmosphere, but rather the velocity of the ray when it reaches the retina. However, be the quantity of this acceleration ever so great, since from the most accurate observation it appears that the *quantity* of aberration is not increased in consequence of the increased velocity of the ray, it follows that

these bodies must also impart to light a velocity in the direction of the earth's motion ∻l to the increase of velocity which they produce.—*See* note (*b*), Chap. II. Book I.

The motion of the planet about the earth in the time in which light comes from the planet to the earth is the whole aberration; ∵ if 1 : r represent the ratio of the sun's distance from the earth to the planet's distance from earth, we have $8', 7''. r$ for the time light takes to come from planet to earth, and if m be the diurnal motion of the planet we have the aberration of the planet $= \frac{8', 7''. r.m}{24^h}$; for the sun the aberration is nearly constant, in order to get the true place we should add 20″, 25 to the place, as given in the tables.

As it is very probable that our planetary system has a motion in space, there must result from it an aberration in the fixed stars, which depends on their situation with respect to the path described by the system; however as the direction of this translation, and also its velocity are unknown, the aberration which results from it is confounded with that arising from the *proper motions* of the stars, so that the coefficient a does not arise *solely* from the velocity of light, combined with the motion of the earth.

Since the distances and magnitudes of all the bodies composing the planetary system are determined relatively to the distance of the earth from the sun, it is of the last consequence that this base should be determined as accurately as possible; this is the reason why the problem of finding the distance of the sun from the earth has occupied so much of the attention of astronomers.

If the annual parallax amounted to 6″, in a triangle of which the vertix is the angle at the star = to 3″, and whose subtense is half the diameter of the earth's orbit, the distance of the star will be expressed by 212,207, the radius of the earth's orbit being unity; and as the radius is 24,096 times the semidiameter of the earth, the dis-

tance of the star from the earth = 5113339872 terrestrial radii, *i. e.* more than five trillions of leagues.

CHAPTER III.

(*a*) As the top and bottom of the tower are supposed to describe, during the fall, similar arcs, and as the body when it arrives at the ground is as far from its first position, as the top of the tower is from its first position. (If the experiment be supposed to be instituted at the equator, and in a vacuo) we have from similar triangles, dividendo, the deviation to the east = to the height of the tower multiplied into the arc described by the bottom, and divided by the radius of the equator, but as the earth revolves uniformly, the arc described varies as the time, *i. e.* as the square root of the height, ∴ the deviation varies as $h \times h^{\frac{1}{2}}$, *i. e.* as $h^{\frac{3}{2}}$, in any latitude ψ the arc described is to the arc described at the equator as cos. ψ : 1 —Mechanique Celeste livre 10, chap. 5.

(*b*) See Notes to Chapter I.

(*c*) This is called the motion of translation; it supposes that each element of the earth has a motion = and parallel to that of the centre, and consequently that the resultant of all the motions is equal to the sum of the motions of the elements. And as all the particles or elements are equally affected by this motion of translation, it cannot affect the rotation of the whole about an axis. The double motion of the earth may result from one sole impulse. The axis of the earth's rotation is not strictly speaking always parallel to itself, for the phenomena of precession and nu-

tation arise from slow motions in the equator, which necessarily implies a motion in the axis.—*See* Note (*d*) to page 280.

(*d*) On the supposition that the earth was immoveable, the change of seasons and different lengths of days were produced by the sun, ascending or descending from one tropic to another; on the hypothesis of the earth revolving on its axis, it presents itself to the sun under different aspects in different parts of its orbit; in both cases, the different lengths of the day and of the seasons, depend on the latitude of the place and declination of either the sun or earth; one of those being = and of a contrary denomination with the other.

(*e*) The orbits being supposed to be circular, or the velocity being that of a planet at its mean distance, we have $\frac{r}{p^2} = \frac{v^2}{r}$, but p^2 is as r^3, $\because v^2 \backsim \frac{1}{r}$, or $v \backsim \frac{1}{\sqrt{r}}$.

(*f*) For in this case the motion being directed either from or towards the earth, it is evident the planet will appear relatively to the earth to be stationary.

(*g*) If lines be supposed to be drawn from different points of the earth's orbit to a star situated in the pole of the ecliptic, they will constitute a conical surface, of which the summit is the star, and the base the orbit of the earth, and the production of this surface beyond the summit, will form another cone opposite to the first, the intersection of which with the celestial sphere will be an ellipse, in the circumference of which the star will always appear diametrically opposite to the earth, in the continuation of a ray drawn from it to the summit of the cone; this circumstance sufficiently distinguishes the effect of annual parallax from that of aberration, which affects the apparent position of the star perpendicularly to the radius of the earth's orbit and not in its direction; the centre of the ellipse is the true place of the star, its greater axis = the parallax, and the minor = the parallax × into the sine

sine of the stars latitude, and it exists in the plane of a circle of latitude passing through the pole; this ellipse is therefore different from that which is described in consequence of aberration; however though the two causes act at once, it would not be difficult to prove that a star under the influence of both would still appear to describe an ellipse about its true place.

Let $c =$ rad. of the earth's orbit, b the distance of star from plane of the ecliptic; $a, a' =$ the curate distances of the star from the sun, and earth; β, β', the heliocentric and geocentric latitudes of the star, α the distance of earth from syzygies, e distance of star from sun; tan. $\beta = \frac{b}{a} = m$, and $a' = \overline{a^2 + c^2 + 2ac. \cos. \alpha}|^{\frac{1}{2}}$, let $\frac{c}{a} = n$, $\frac{c}{e} = p$, then tan, $\beta' = \frac{b}{a'} = \frac{m}{\sqrt{1 + 2n. \cos. \alpha + n^2}} =$ (neglecting n^2 which is inconsiderable) $m.(1 - n. \cos. \alpha)$, $\therefore$ tan. $(\beta - \beta') = \frac{mn. \cos. \alpha}{1 + m^2.(1 - n. \cos. \alpha)}$, *i. e*, $\beta - \beta' = \frac{m.n. \cos. \alpha}{1 + m^2} = n.$ $\cos. \alpha. \sin. \beta. \cos. \beta$, $\because$ as $\cos. \beta = \frac{a}{e}$, and $p = n. \cos. \beta$, $\beta - \beta'$ the parallax in latitude $= p. \cos. \alpha. \sin. \beta$; note p is the semidiameter of the orbit of the earth as seen from the star, and $\because$ it is $=$ to the annual parallax. The tangent of the angle formed by lines drawn from projection of star on the plane of the ecliptic to sun and earth, or the parallax in longitude $= \frac{c. \sin. \alpha}{a + c. \cos. \alpha} =$ $\frac{p. \sin. \alpha}{\cos. \beta + p. \cos. \alpha}$, *i. e.* the parallax in longitude $= \lambda' =$ $p. \sin. \alpha. \sec. \beta$. very nearly; consequently, $\lambda' : \beta - \beta' ::$ $\tan. \alpha : \sin. \beta. \cos. \beta :: 2 \tan. \alpha : \sin. 2\beta$, hence we can determine the one from the other; $\beta - \beta'$ vanishes in the quadratures, *i. e.* when a $\alpha = 90$ or 270; it is a maximum in the syzygies; in this particular it differs from the aberra-

tion; its maximum being p. sin. β, it is greatest near to the pole of the ecliptic, β—β, is positive, or the apparent latitude is less than the true from the last quadrature through conjunction to the first quadrature; in the other half of the orbit it is negative, or the apparent latitude is greater than true, λ' vanishes in the syzygies, and it is a maximum and $= p$. sec. β, in the quadratures, it $\because$ increases with the latitude, and from conjunction to oppósition it is positive, or the apparent latitude is greater than true, and from opposition to conjunction it is less than true; the apparent latitude in opposition $= m.(1 - n$ cos. $a)$, and is a minimum; it is $= m.(1 + n.$ cos. $a)$ in conjunction, when it is a maximum.

If $\triangle$ be the difference between the longitude of a star in the 90th and 270th degrees of distance from conjunction, we have $\triangle = 2p$. sec. β, $\because p = \frac{\triangle}{2}$. cos. β.

If $p = 20'' = a$, the same tables would serve for parallax and aberration, if they are computed for the aberration it is only necessary to add 90° to the sun's place.

CHAPTER IV.

(a) The locus of a planet and consequently its orbit, which is composed of all its points, is determined by the magnitude of the radius vector and by the angle which it makes with some line fixed in space, such is that drawn to the first point of Aries. With respect to the *direction* of the radius vector, this is found by observing the planet in opposition or conjunction; for in this case, on account of the

irrationality which exists between the period of the earth and planet, they occur in different points of the orbit, consequently, we can by means of oppositions and conjunctions, find all the points of the orbit, and also the epoch, when the planets are in those positions; hence, may be obtained the law which exists between the heliocentrick longitude and time, from which may be derived the true longitude; and as the principal inequalities are destroyed at the termination of each revolution, this law may be developed in a series, proceeding according to the sines of angles $\div$l to the time and their multiples; the coefficients of this series may be determined by observations made under the most favourable circumstances.

(*b*) See Notes to page 12.

(*c*) In order to determine the magnitude of the radius vector, the observations made at quadratures are the most useful, for the radius being then perpendicular to the visible ray, it appears under the greatest angle; and as the quadratures occur in every point of the orbit, the law between the time and radius vector, and $\therefore$ between this last and the longitude can be determined, $\therefore$ the orbit can be completely constructed; in case of an inferior planet, the greatest elongations are employed in place of the quadratures to determine the radii vectores.

(*d*) Let ψ be the arc described about the sun, r the distance of planet from sun, then the angular velocity $= \frac{d\psi}{r}$ is observed to be equal to $\frac{A}{r^2}$, $\therefore r.d\psi =$ twice the sector described in an indefinitely short period of time $= A$.

To completely determine the orbit of a planet, 1st, the plane in which it moves—2dly, the nature of the curve described—3dly, the position of this curve in the plane of its orbit, aad 4thly, the law according to which this curve is described, must be determined; the law is given by the application of Kepler's 2d law, the position by that of its

greater axis; the species of the curve by Kepler's 1st law; the particular form by the excentricity, the *magnitude* of the axis, by the revolution or mean motion—which last, as determined by a comparison of ancient and modern observations is the best known of all the elements.

(*f*) The period may be found by noting the time between two returns of the planet to the same node, and this interval being divided by the number of revolutions, will give the period with respect to the node; but as this node regrades, the period thus deduced will be less than the true period; however P may be easily computed from knowing the quantity of regression a, for if n be the number of revolutions, we have $n.360-a : 360 ::$ observed time: P. The period may be also found from the formula given in Notes, page 323, for $P = \frac{p.t}{p+t}$; (t being the time between two conjunctions and oppositions). The axis major or mean distance can be determined by means of Kepler's third law; the earth's orbit and period being accurately known already. To determine the excentricity, let the heliocentric positions of the planet, when the equation of the centre is observed to be a maximum, *i. e.* when the planet is moving with its mean angular velocity, be determined; the mean places of the planet at these epochs can be determined, and they always lie between the perihelion and the true places; ∵ the angle at the sun formed by lines drawn to the true places are given by observation, and the time between the two observations gives the angle at the sun formed by lines drawn to the mean places; the difference between these angles = twice the greatest equation; as the points when the true and mean motions are the same, are not exactly known, among a great number of observations, those two should be selected which give the difference between the preceding angles the greatest possible, we may then as-

sume their difference equal to twice the greatest equation, as near to the maximum, this variation is inconsiderable.

The excentricity is given, from the greatest equation, by means of the series—(*see* Notes to page 14.)

$e = \frac{1}{2} h - \frac{11}{768} h^3 - \frac{587}{983040} h^3$, &c. e represents the excentricity; $h = \frac{g}{57.29578}$ g expressing the greatest equation of centre.

If the planet be observed near the aphelion, the difference between angle proportional to the interval from the planets being in the point where the equation of the centre is a maximum, to the time when the planet is in aphelion, and the angle between axis major and line drawn to this point, should be = to the greatest equation, as it is next to impossible that this should be accurately the case, let is be less by a small angle c, and make a second observation when it is greater by an angle c'; now as the longitudes of the planet when observed at each side of the aphelion, and ∴ their difference e are known, and also t the interval between the observations, we have when the angles are very small, $c + c' : e :: c$ to the angular distance of the first assumed point which is known, from the aphelion, we have also $q.p : c + c' : c :: t$: to the *time* from this point to aphelion, which ∴ determines its epoch; ∴ we can obtain the longitude at any epoch, or *vice versa*.

Let L, l represent the heliocentric longitudes of the sun and node, S the angle at the sun subtended by the earth and planet = L—l, E the elongation of planet from sun = difference between the geocentric longitudes of sun and planet; then r the planet's distance from sun : R the earth's distance sin. E ; sin. (S + E), ∴ r. sin. (S + E) = R. sin. E; *i. e*, r. sin. (E + L—l) = R. sin. E; let E′, R′, L′, be the values of E, L, R, when the planet returns again to the node, then r. sin. (E′ + L′—l') = R′. sin. E′,

$$\therefore \frac{\sin.(E' + L' - l') + \sin.(E + L - l)}{\sin.(E' + L' - l') - \sin.(E + L - l)}$$
$$= \frac{(R'. \sin. E' + R. \sin. E)}{R'. \sin. E' - R. \sin. E}, \because$$
$$\frac{\tan.(\frac{1}{2}(E' + E + L' + L) - l)}{\tan.\frac{1}{2}(E' - E + L' - L)} = \frac{R'. \sin. E' + R. \sin E}{R'. \sin. E' - R. \sin. E};$$

hence as R', R, E', E, L', L, can be determined, we can find l the longitude of the node; this method supposes the planet to be in its node, if not, let β, β', be the geocentric latitudes of the planet before and after its passage through the node, t the interval between the observations, then $\beta + \beta' : \beta' : t$ to the interval between the first observation and the time when the planet is in the node; hence we can find E and L when the planet is in the node. This method supposes also that the node is stationary, which is not the case, (*see* Chapter III. Vol. II.) However a determination of the node in this manner will give the motion of the node, by means of which l can be determined accurately; the inclination i is easily determined, for we have $\sin. E = \tan. \beta. \cot. i$.

The preceding methods not being rigorously exact, the elements determined by means of them will be found to differ somewhat from the truth; their values should be corrected by the formation of equations of condition, of which the number is indeed indeterminate; it is only necessary to have as many of them as there are unknown quantities to be determined.

(*f*) On the secular inequalities *see* Notes to Chapter II. Vol. II.

(*g*) The reader is likewise referred to Chapter II. for an explanation of the variation in Jupiter's and Saturn's motions.

An inspection of the axes majores, or mean distances of the ancient planets, shews that, with one exception, their distances are embraced in the formula $4 + 3.2^{n-2}$, (n being the place occupied by the planet; commencing with mercu-

ry, however a blank occurred between Mars and Jupiter; and in order to have the preceding law exact, a planet should exist at the distance where the four new ones have been observed. The circumstance of there being four instead of one planet at this distance, does not militate against the preceding law, as from some circumstances connected with them it has been conjectured that these might originally have constituted but one planet.—*See* Notes, page 333, and Vol. II. Chapter II.

More particularly, the causes which disturb the motions of the four new planets arise from their orbits mutually intersecting each other, from their comparatively great excentricities, and from the proximity of Jupiter, the greatest of all planets.

CHAPTER V.

(a) Let a, b, represent the major and minor semiaxes of the ellipse A, P the periodic time, s the sector described in any time t, a', b', A', P', s', corresponding quantities for another ellipse, then since the areas are ∻l to the times, we have $s = \frac{A.t}{P}$, and $s : s' :: \frac{A}{P} : \frac{A'}{P'}$

but $A = ab$, $A' = a'.b'$, and $P = Ka^{\frac{3}{2}}$, $\therefore s : s' :: \frac{a.b}{K.a^{\frac{3}{2}}}$

$\frac{a'.b'}{K.a'^{\frac{3}{2}}}$.

(b) Let x be the perihelion distance, and we have $b^2 = x.(2a - x)$, $\therefore s : s' :: \frac{\sqrt{x.(2a-x)}}{\sqrt{a}} : \sqrt{x}$; the ellipse A'

being supposed to become a circle of which the rad. $=x$; $s : s' :: \sqrt{2a-x} : \sqrt{a}$, which when the ellipse A becomes a parabola, in which case, x vanishes relatively to a, the proportion becomes that of $\sqrt{2} : 1$; the ratio of the sector described by the fictitious planet to the synchronous sector described by the earth at a distance from the sun equal to r, is that of $\sqrt{x} : \sqrt{r}$; $\therefore$ we can determine for any instant whatever the area traced by the radius vector of the comet, commencing with the instant of its passage through the perihelion.

The time t of describing a sector $s = \frac{P.s}{A} \backsim \frac{a^{\frac{3}{2}}.s}{a.b} \backsim \frac{s}{\sqrt{p}}$, p being the parameter; hence it appears that the times in different sectors, are as the sectors described divided by the square root of the parameters.

(c) In orbits of great excentricity, such as the comets, the equation $r = \frac{a.(1-e^2)}{1+e.\cos. v}$, may, by substituting $1-a$ for e, be made to assume the form

$$\frac{D}{\cos.^2\frac{1}{2}v.\left(1+\frac{a}{2-a}.\tan.^2\frac{1}{2}v.\right)},$$

(D being the perihelion distance.) For as $\cos.^2\frac{1}{2}v+\sin.^2\frac{1}{2}v=1$, and as $\cos. v = \cos.^2\frac{1}{2}v - \sin.^2\frac{1}{2}v$, we have $r =$

$$\frac{a.a.(2-a)}{\cos.^2\frac{1}{2}v+\sin.^2\frac{1}{2}v+(1-a).(\cos.^2\frac{1}{2}v-\sin.^2\frac{1}{2}v)},$$

$=$ by concinnating and dividing by $(2-a)$,

$$\frac{a.a}{\cos.^2\frac{1}{2}v\frac{+a}{2-a}\sin.^2\frac{1}{2}.v.}$$, and as $D = a.(1-e)$ $=a.a$, we $\therefore$ obtain $r=\frac{D}{\cos.^2\frac{1}{2}v.\left(1+\frac{a}{2-a}.\tan.^2\frac{1}{2}v\right)}$, by

expanding this expression into a series we obtain r to any degree of accuracy; if a vanished the expression would become $\frac{D}{\cos.^2\frac{1}{2}v}$; the time corresponding to the true anomaly in an orbit, such as the preceding, may be likewise found, for as $u = 2\tan.\frac{1}{2}u.(1-\frac{1}{3}.\tan.^2\frac{1}{2}u+\frac{1}{5}.\tan.^4\frac{1}{2}u$, &c.) and as $\tan.\frac{1}{2}u = \frac{\sqrt{1-e}}{\sqrt{1+e}}.\tan.\frac{1}{2}v = \frac{\sqrt{a}}{\sqrt{2-a}}.\tan.\frac{1}{2}v$, by substituting we obtain $u = \frac{\sqrt{a}}{\sqrt{2-a}}.\tan.\frac{1}{2}v.\left(1-\frac{1}{3}\left(\frac{a}{2-a}\right).\tan.^2\frac{1}{2}v+\frac{1}{5}.\left(\frac{a}{2-a}\right)^2.\tan.\frac{1}{2}v-\&c.\right)$; but $\sin.u = \frac{2\tan.\frac{1}{2}u}{1+\tan.^2\frac{1}{2}u} = 2\tan.\frac{1}{2}u.(1-\tan.^2\frac{1}{2}u+\tan.^4\frac{1}{2}u,\&c.)$, $e.\sin.u = 2(1-a).\frac{\sqrt{a}}{\sqrt{2-a}}.\tan.\frac{1}{2}v.\left(1-\frac{a}{2-a}.\tan.^2\frac{1}{2}v+\left(\frac{a}{2-a}\right)^2.\tan.^4\frac{1}{2}v,\ \&c.\right)$. ∴ in the equation $nt = u - e.\sin.u$, the substitution of these values of u and of $e.\sin.u$ will give t in a very converging series, in a function of the anomaly v, and $= \frac{1}{n}.\tan.\frac{1}{2}v.\left(1+\frac{(\frac{2}{3}-a)}{2-a}.\tan.^2\frac{1}{2}v-\frac{\frac{4}{5}-a}{(2-a)^2}.a.\tan.^4\frac{1}{2}v+\&c.\right)$, which when $a = 0$, $\frac{1}{n}.\tan.)\frac{1}{2}v+\frac{1}{3}.\tan.^3\frac{1}{2}v.)$

If $v = 90$, then $\tan.\frac{1}{2}v+\frac{1}{3}.\tan.^3\frac{1}{2}v = \frac{4}{3}$; and t' the time corresponding to this anomaly $= \frac{4}{3.n} = 109^d, 6154$, when $D=1$; ∴ a comet, of which the perihelion distance $= 1$, will describe in this time a sector of which the anomaly is 90, *i. e.* it will reach the parameter in that time, ∴ for any other anomaly u, we can obtain the corresponding time; the determination of the anomaly from knowing the time is more difficult than the reverse problem, for u must be determined by an equation of the third degree.

Note.—This is called the comet of 109 days, and for any time t' we have tan. $\frac{1}{2}v$+3. tan. ${}^{3}\frac{1}{2}v = \frac{t'}{27,40385}$, and for that of which the perihelion distance $= x$, tan. $\frac{1}{2}v$ + 3. tan. ${}^{3}\frac{1}{2}v = \frac{t}{27,40385x^{\frac{3}{2}}}$; hence, if the comets move in parabolas, their anomalies depend only on their perihelion distance.

The formula for determining the time of describing *any* arc intercepted between the radii vectoris r, r'. is $T = \frac{T}{12\pi}.\left((r+r'+c)^{\frac{3}{2}} \pm (r+r'-c)^{\frac{3}{2}}\right)$.—*See* Celestial Mechanics, Book II. Chapter IV.

(c) In consequence of the smallness of the diameter of a comet, and the feebleness of its light, it does not become visible until it approaches very near to the sun, so that the greater number of comets which have been observed, appear nearer than Mercury, shortly after their distances become so great that they cease to be seen ; ∵ their orbits are extremely excentric ellipses, in which particular they differ from the planetary orbits, and likewise in the circumstance that they are inclined at every species of angle to the ecliptic, from which it follows, that their motions are sometimes retrograde; though they receive their light from the sun, their disk is not so accurately terminated as the planetary disks, nor are there any apparent phases; indeed the side averted from the sun appears to be luminous likewise.

The great inclination to the ecliptic is not a distinguishing property of comets, neither is the feebleness of their light or the smallness of their masses, as in all these particulars they do not differ from the planets recently discovered.

The method of determining the elements of the planetary orbits is not applicable to comets which are visible only in a small portion of its orbits; ∵ the most impor-

tant elements, namely, the mean distance and the mean motion cannot be thus determined, it is necessary, in order to obtain them, to avail ourselves of Kepler's laws.

In the methods made use of for determining the planetary orbits, it is assumed that the planet has been observed more than once in the same point of its orbit, from which the periodic term and distance from the sun can be determined. The sun being assumed to be in the focus of the ellipse or parabola, which the comet is supposed to describe, if the comet be observed in three different positions from three corresponding points of the earth, in the triangle formed by lines joining the sun, earth, and comet; we only know the angle of elongation at the earth, and the distance of the earth from the sun, which is not enough; however, in the two triangles formed by lines drawn from the sun to the observed places of the comet, we have not only the *ratio* of their areas from knowing the times between the respective observations, but also the areas themselves, the conic section described being supposed to be known, and by combining these data we can determine the orbit.

In this determination an indirect method is generally employed as less complicated, and as more exact than the direct determination of the elements, on account of the errors of observation. In this way two of the unknown quantities are assumed arbitrarily, by combining them so as to satisfy one of the observations, with those elements, the other observations are calculated hypothetically, and then a comparison of the computation with the observations will indicate the correction required for the elements. Now, as the great excentricity of the orbit justifies us in assuming that the orbit is q. p. parabolic; there is also this peculiar advantage in assuming them to be such, namely, that the : :nality of the areas to the times is reduced to the quadrature of the curve, which, in

the case of the parabola is extremely simple; besides as all parabolas are similar curves, we can compute a general table for all orbits.

Several indirect methods have been proposed for determining the cometary orbits on the parabolic hypothesis, and they only differ from each other in the elements which are supposed to be known. The following is a brief outline of the method which supposes the angle at the sun to be known.

By a comparison of two geocentrick positions reduced to the ecliptic, and by assuming the corresponding angles of commutation arbitrarily, we can compute by means of *these* angles, and of the given elongations and distances of the earth from the sun at the times of the two observations, the curtate distances of the comet from sun at these times, and also the heliocentric movement on the ecliptic, or the angle contained between these distances; from knowing the angles at the earth and sun, and also the geocentric latitudes, we can determine the heliocentric latitudes, and also the true distances of comet from sun at the times of observation. With the heliocentric latitudes and longitudes we can determine the inclination, the position of the node, and the longitudes on the orbit; ∵ we have two radii vectores, and the angle contained between them; hence we can determine from the nature of the parabola, the perihelion distance, the longitude of the perihelion, the area of the sector contained between the radii, the time employed in moving from perihelion to the observed places, from which we can determine the instant of the passage through the perihelion; if the time *computed* for passing from one observed position to the other, does not agree with the time elapsed between the two observations, the *assumed* angles of commutation do not take place simultaneously; ∵ one should be changed until the computed time agrees with the observed, the other remaining the same; now all the elements of the orbit

being determined, we can calculate for the time of the third observation, the true anomaly conformably to the parabolic hypothesis, and consequently the longitude on the orbit and the distance from the sun at this time, then from knowing the position of the node, and the inclination, the heliocentric longitude and latitude, and also the curtate distance of the third point from the sun may be determined; the longitude of this point and of the earth at the time of the third observation, will make known the angle of commutation at this time. Knowing this angle, and the distances of the earth and comet from the sun, we can compute the angle of elongation, which ought to be equal to the observed angle; likewise the first angle of commutation is also erroneous, ∴ by assigning another value to it, the second commutation will be changed until the first and second observations agree with the computation; we should operate on the third observation in the preceding manner, and if it does not agree with the computation, the first angle of commutation should be again changed. After thus making two hypotheses for the first angle of commutation, their errors will indicate by the method of interpolations the correction to be applied to this angle, in order that the hypothesis should satisfy the three observations. With those elements we can reduce any observation to its heliocentric position, from which it is easy to calculate with the true anomaly the time of any observation, which enables us to verify the elements by all the observations which have been made, and to correct them by taking the mean.

The element which in the case of the planets is the first and easiest to be determined, namely, the periodic time, is in the case of the comets the last and most difficult, and cannot be found except by a computation on the hypothesis that the orbit is elliptical.—*See* Celestial Mechanics, Book II. Chapter IV. and Delambre, tom. III. Chapter XXXIII.

If, as stated in page 197, the elements of a comet nearly agree with those of a comet formerly observed, we can apply the calculus of probabilities to determine to what degree of probability we can be sure that they are exactly the same.

(*d*) The heat of the sun is as the density of his rays, *i. e.* inversely as the square of the distance; now the heat of boiling water is three times greater than that produced by the action of the sun in summer on the earth; and iron heated to a red heat is four times greater than that of boiling water, therefore the heat which a body of the same density as our earth would acquire at the perihelion distance of the comet, is at least 2000 times greater than that of iron heated to a red heat; and it is quite evident that with such a heat, all vaporous exhalations, and in fact every species of volatile matter ought immediately to be dissipated; the preceding is Newton's estimation, *see* Princip. Math. Book III. page 509; he assumes that the comets are compact solid substances like the planets; this he infers from their passing so near to the sun in their perihelion without being dissipated into space.

Heat expands all bodies, but = additions of caloric do not produce equal increments of magnitude, for as it acts by diminishing the cohesive tendency, the greater that tendency the less will be its effect; on the contrary, in the case of gases, as no such tendency exists = increments of heat must necessarily produce equal augmentations of bulk. In general, when the density of bodies is increased they must give out caloric. The quantity given out by water when freezing is 140°, its capacity is by this increased one-ninth; from this it has been inferred, that the zero of temperature is 1260 degrees below the freezing point; but there are great discrepancies in the results from different liquids.

The latent heat of the vapours of fluids, though con-

stant for vapour of the same kind and of a given elasticity, still varies in different vapours; thus, according to a recent investigation, the vapour of water at its boiling point = 967°. However, though this heat is different in different fluids, still the point at which all solid bodies, and all those liquids which are susceptible of ignition, *i. e.* of becoming heated so as to be luminous *per se*, is nearly the same for all, and about 840° of Fahrenheit.

In permanently elastic fluids, the caloric is held so forcibly that no diminution of temperature can separate it from them.

The comet of 1770 is the only one which cannot be computed on the hypothesis that it moves in a parabola.—*See* Vol. II. Chap. IV. Notes.

The nebulosity which environs the comet is its atmosphere, which extends farther than our atmosphere; it increases according as it approaches the sun. The parts which are volatilized become so very light, that the attraction of the comet on them is nearly insensible, so that they yield without difficulty to the impulsion of the solar rays; the orbit described by each particle must be an hyperbola, for previously to the impulsion, as it described a parabola, its velocity is to the velocity in a circle at the same distance as $\sqrt{2} : 1$, and the impulsion of the solar rays increasing this velocity, it will be to the velocity in a circle in a greater ratio than that of $\sqrt{2} : 1$, it must consequently describe an hyperbola.

The tail is generally behind the comet; this is the cause of the curvature which has been observed in it, and also of the deflection towards that part from which the comet is moving.

It has been supposed that the loss sustained by the evaporation near the perihelion may be repaired by new substances which it meets with in its route.—*See* Chapter VI. Book V. Vol. II. Notes.

CHAPTER VI.

The elements of the orbits of the satellites in the order in which they are derived, the one from the other, are the periodic time, or mean motion, the distance from the primary, the inequalities and true motion, the inclination, and nodes, and magnitude.

In determining the period from the interval between two consecutive conjunctions, we obtain it as affected by all the inequalities in the motions of the satellites; but when it is obtained from two conjunctions, separated by a considerable interval from each other, these inequalities are in a great measure compensated. Observations with the micrometer give, as was stated in page 96, the angle which the radius of the orbit subtends at the earth, it must change with the distance of Jupiter from the earth; but as the apparent diameter of Jupiter varies in the same ratio, it is only necessary to measure this diameter at the same time, in order to have the diameter of the orbit relatively to that of Jupiter; and as a comparison of these diameters at different times gives this ratio always the same, it follows that the orbit is q. p. circular. The distances of the satellites might also be inferred from the greatest durations of the eclipses, and *vice versa.* Some of the observed inequalities are only apparent, others are real; if there is a difference in the *periodic* revolutions of the satellites, it must arise from a real inequality in the motion of the satellite; but as the synodic revolution depends on the motion of Jupiter, there may be a difference in the observed synodic revolutions, without there being any inequality in the satellite from which it may have originated. When the computed time of an eclipse

is corrected for the inequalities in the motion of Jupiter, and also for the velocity of light, &c., then a comparison of this time, with that furnished by observation, will enable us to discover the real inequalities.

The cause of the deviations from mean motion arise either from the excentricity of the orbits, or from the disturbing action of Jupiter combined with that of the sun: the disturbing action of the satellites on each other depends on their relative positions; its period therefore will be the time at the end of which the satellites return to the same relative position, with respect to the sun; and as the eclipses are the most important observations, and those most commonly made, it is therefore the period in which each satellite makes a complete number of revolutions; but a comparison of the values given in page 208, shews that the shortest period which satisfies these conditions for the *three* first satellites is 437 days. This period is less exact with respect to the fourth satellite, as it performs in 435 days 26 revolutions; however as its actions are less than that of the other satellites, on account both of its greater distance and smaller mass, and as the difference does not exceed one day and a half, it is assumed that even with respect to it, the period is 437 days. Astronomers made use of this period to form empirical equations, for which those founded on the theory of universal gravitation have been substituted. Their arguments are composed of the position of each satellite with respect to the others, the apsides of the third and fourth, and the nodes of their orbits.

The orbits are unquestionably elliptic, however the ellipticity of the two first satellites cannot be observed. The eclipses will be observed sooner when the planet is in its perijove, and later in the apogove, than the computed time, which will enable us to determine the position of the apsides. If there was no *penumbra*, and if the diameter of the satellite was insensible, the duration of the com-

puted and observed eclipses would be the same; but as these causes affect the observed time of commencement, it is evident that it depends on the eye of the spectator, and also on the goodness of the telescope.

The tables are so constructed as to give the eclipses in the mean state of the atmosphere, mean power of the telescope, and mean accuracy of vision; besides what is mentioned in page 155, the proximity of the star to the horizon, its proximity to Jupiter, or Jupiter's too great proximity to the sun; all, or any of these circumstances affect the observations. In order that the results given by stationary observers should agree with those given by voyagers, we should employ only telescopes of a medium magnifying power.

At the extreme distance from the node at which an eclipse can happen, the duration of an eclipse is the *least* possible, and would be always the same if the inclination was constant; but as this duration is variable, for the 1st, 2d, and 3d satellites particularly, it follows that the inclination is likewise variable.

The position of the node will be given from knowing the duration of the longest eclipse; the shortest observed eclipses are at the limit, and will give the inclination; knowing the position of the node and inclination we can compute antecedently the duration of any eclipse.

Calling M, M', M'', the mean motions of the three first satellites, and l, l', l'', their mean longitudes; we have also $M+2M''=3M'$, and $l+2l''=3l'+180°$; these equations are so exact, that the deviations from them, which are observed, must arise from errors of observations, or from the small oscillations which they make about these mean values, *see* Vol. II. Chap. V.

It follows from this, that these three satellites cannot be simultaneously eclipsed, for then we would have $l=l'=l''$; or $l+2l''=3l'$, which, in consequence of the second equation, is impossible; and it appears from the first equation,

that if the first is true once, it will be always so; it likewise follows, that the real inequalities of those three satellites must have precisely the same laws and periods.

The method alluded to in page 328 would evidently give a diameter, as seen from Jupiter, smaller than the actual magnitude. It has been suggested, that if in geocentric conjunctions of the satellites with Jupiter, the instants of interior and exterior contact with Jupiter were observed at immersion and emersion, we would have the time which the planet takes to describe a chord equal to its diameter; this will give the ratio of the diameter of the satellite to that of Jupiter, if that observation in which the ratio of the duration of the passage to that of the immersion is the greatest possible, be observed.

BOOK THE THIRD.

CHAPTER I.

(*a*) THAT which admits of the introduction of a finite body has been called *space*; it is said to be pure if it be totally devoid of matter. Whether there be such a thing as any space absolutely pure has been disputed, but that such a space is *possible*, admits of no dispute; for if any body be annihilated, and all surrounding bodies kept from rushing into the space which this body occupied, that portion of space, with respect to matter, would be pure space. Pure space is therefore conceivable, and it is conceived as having length, breadth, and depth. In the notion of motion, as announced in the text, the author assumes that there would be motion even though all the other bodies in the converse were annihilated, but this position is not acceded to by all philosophers. Berkeley, for instance, thought that all motion was relative; however, though with respect to the origin of our ideas of motion, his account is unanswerable; nevertheless it must be admitted, that a body might spontaneously produce motion in itself; still we may venture to affirm with him, that as long as the body would remain in absolute solitude it would not acquire the idea of motion; but if other bodies be

called into existence, while the body is under the influence of its own spontaneous energy, it certainly would then acquire the idea of motion, from perceiving its change of place with respect to those bodies; but as this creation of bodies at a distance could produce no real alteration in the condition of a body which existed before them, if the body *now* perceives itself to be moving, we may conclude that it was moving previously to the existence of those bodies, and that its motion was *absolute*.

(*b*) All cases of the equilibrium of forces acting on a material point, may ultimately be reduced to that of two equal and opposite forces, as when any number of forces acting on the same point constitute an equilibrium, all of them but one may be reduced to a force equal and contrary to this one, so that these forces are always as the sides of a polygon, having the same number of sides drawn parallel to their directions. (Note, the sides of the polygon are not necessarily in the same plane.)

If three forces acting on a material point constitute an equilibrium, they must exist in the *same* plane; four forces acting in *different* planes constitute an equilibrium, when they are as the three sides and diagonal of a parallelopiped respectively parallel to their directions. If two equal and parallel forces act in opposite directions, an equilibrium between them cannot be effected by the introduction of any third force.

(*c*) It is evident from this, that in the composition of forces, force is expended—in the resolution force is gained. The two given forces into which the given one is resolved are reciprocally as perpendiculars from the given force on the directions of its components. The less the angle made by the components, the greater will be the resultant, therefore it is a maximum when this angle $= 0$, *i. e.* when the components are parallel; in this case it is easy to prove that the resultant $=$ the sum of the components,

and that its point of application divides the line connecting them inversely as the forces.

(*d*) Any force being resolved into three others, at right angles to each other, as stated in page 225, the line representing it will be the diagonal of a rectangular parallelopiped, of which the composing forces represent the sides, $\because$ A, B, C, representing the composing forces, $\sqrt{A^2+B^2+C^2}$ will represent the resultant or diagonal; and

$$\frac{A}{\sqrt{A^2+B^2+C^2}},\ \frac{B}{\sqrt{A^2+B^2+C^2}},\ \frac{C}{\sqrt{A^2+B^2+C^2}},\ =$$

the cosines of the angles which A, B, C respectively make with $\sqrt{A^2+B^2+C^2}$ it is also evident that the sum of their squares $=1$; if A', B', C' be the components of a second force parallel to the same rectangular coordinates, the coordinates of the resultant of $\sqrt{A^2+B^2+C^2} = S$, and of $\sqrt{A'^2+B'^2+C'^2}=S'$, are $A+A'$, $B+B'$, $C+C'$, respectively, therefore as these are the coordinates of the diagonal of a parallelogram whose sides $= \sqrt{A^2+B^2+C^2}$ $\sqrt{A'^2+B'^2+C'^2}$, this diagonal must be the resultant of the given forces S, S′, and if the angle between their directions $=\triangle$, we have $S^2+S'^2-2\,S.S'.\cos.\triangle=(S.\cos.a-S'\cos.b)^2+(S.\cos.a'-S'.\cos.b')^2+(S.\cos.a''-S'.\cos.b'')^2 = S^2+S'^2-2SS'.(\cos.a.\cos.b+\cos.a'.\cos.b'+\cos.a''\cos.b'')$, therefore $\cos.\triangle = \cos.a.\cos.b+\cos.a'.\cos.b'+\cos.a''.\cos.b''$.

Note a, a', a'', b, b', b'', are the angles made by S, S′ with the rectangular coordinates. The value of cos. $\triangle = 0$, when S, S′ are at right angles to each other; as $A+A'$, $B+B'$, $C+C'$, are the coordinates of the resultant of S and S′, $A+A'+A''$, $B+B'+B''$, $C+C'+C''$, are the coordinates of V, the resultant of S″, and this last resultant, $\because$ V will be as stated in the text, the diagonal of a parallelopiped, whose sides are $A+A'+A''+$ &c.; $B+B'+B''+$ &c., $C+C'+C''$ &c.; $V^2=(A+A'+A''+\&c.)^2+(B+B'+B''\&c.)^2+(C+C'+C''$

$+$ &c.)^2 and if $m\ n\ p$ be the angles which V makes with the axes, we have $\cos. m = \frac{A+A'+A''+\ \&c.}{V}$, $\cos. n = \frac{B+B'+B''+\&c.}{V}$, $\cos. p = \frac{C+C'+C''+\&c.}{V}$, $\therefore$ we have both the quantity and direction of the resultant.

The coordinates of the origin of the force S being supposed to be A B C, if $x\ y\ z$ be the coordinates of its point of application to the given point, the distance of the point of application from the origin, $= s = \sqrt{(x-A)^2+(y-B)^2+(z-C)^2}$ $\therefore$ the force resolved parallel to the coordinates $= S\frac{(x-A)}{s}, S\frac{(y-B)}{s}, S.\frac{(z-C)}{s}$ $= \left(\text{as } \delta x = \frac{\delta s}{\delta x}.\ \delta x + \frac{\delta s}{\delta y}.\ \delta y + \frac{\delta s}{\delta z}.\ \delta z.\right)\ S.\frac{\delta s}{\delta x},\ S.\frac{\delta s}{\delta y}$, $S.\frac{\delta s}{\delta z}$, respectively, in like manner for a second or third force S', S'', $S'.\frac{\delta s'}{\delta x}$ $S'.\frac{\delta y'}{\delta y}$, or $S.''\frac{\delta s''}{\delta x}$, $S.''\frac{\delta s''}{\delta y}$ &c. are the forces S' S,'' parallel to x, y &c. $\therefore$ $\Sigma.\ S.\frac{\delta s}{\delta x}$ is the sum of all the forces S S' S'', resolved parallel to x; now if u be the distance of V the resultant of all the forces S, S', S'', &c. from the given point, $V\frac{\delta u}{\delta x}$ will express the resultant resolved parallel to x, and as by what has been already established, this is equal to the sum of the composing forces parallel to x, we have $V.\frac{\delta u}{\delta x} = \Sigma.\ S.\frac{\delta s}{\delta x}$; $V.\frac{\delta u}{\delta y} = \Sigma.\ S.\frac{\delta s}{\delta y}$; $V.\frac{\delta u}{\delta z} = \Sigma.\ S.\frac{\delta s}{\delta z}$; multiplying these equations by $\delta x\ \delta y\ \delta z$ respectively, we obtain by adding them together V. $\delta u = \Sigma.\ S.\ \delta s$. If S, S', S'', &c. are Algebraic functions of S, S', S'', &c. then $\Sigma.\ S.\ \delta s$ is an exact variation.

(c) The quantity advanced in the direction of the force

is termed its virtual velocity, in the direction of that force. *See* Note (m) page 265.

In the state of equilibrium $V = o$, ∴ $\Sigma.\ S.\ \delta s = o$, ∴ when a point acted on by any number of forces is in equilibrio, the sum of the products of each force by the quantity advanced in its direction is equal to cypher. In this case S one of the forces is = and directly contrary to the resultant V' of all the rest S', S'', S''', &c. for from what has been already stated, we have $V'.\cos. a = S'.\cos. b + S.''\cos. c +$ &c. but since $S.\cos. a + S.'\cos. b + S.''\cos. c +$&c. $= o$, we have $V'.\cos. a = - S.\cos. a$; in like manner it may be shewn that $V'.\cos. l = - S.\cos. a'$, $V'\cos. o = - S.\cos. a''$ ∴ $V'^2 = S^2$; and $a = 180 - a$, $l = 180 - a'$ &c.

(f) If the resultant was not perpendicular to the surface it might be resolved into two forces, one perpendicular to the surface, which would be destroyed by the reaction of the surface, and the other parallel to this surface, which, as it is not counteracted, would cause the point to move on the surface, contrary to the hypothesis. The re-action which the body experiences from the curve or surface is = and directly contrary to the force with which the point presses it; ∴ if R denote this reaction, r being a perpendicular from the point of application to the surface, we must have $o = \Sigma\ S\ \delta s + R\ \delta r$, instead of the equation $o = \Sigma\ S\ \delta s$. If we suppose δx, δy, δz, which are arbitrary, to belong to the surface on which the point is subjected to exist, we have $\delta r = o$; for r is by hypothesis perpendicular to the surface, ∴ $R\ \delta r$ vanishes from the preceding equation, consequently the position of the text is true, or in other words, in the case of the equilibrium of a point, the sum of the forces which solicit it, each multiplied by the space through which the point moves in its direction, is equal to nothing; it ought however to be remarked, that when the point exists on a surface, the equation $o = \Sigma\ S\ \delta s$ is not equivalent to three *distinct* equations, but only to two; for as the variations δx, δy,

δz, belong to the curved surface, one of them may be eliminated by means of the equation of the surface. Laplace substitutes for δr its value N δu, u being the equation of the surface, and N being a function of x, y, z, such that $\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2 = \frac{1}{N^2}$; then if λ be supposed $=$ to N. R, the equation of equilibrium becomes $o = \Sigma . S \, \delta s + \lambda \, \delta u$; in this case we may put each of the coefficients of $\delta x \; \delta y \; \delta z = o$, but still they are only equivalent to two distinct equations, on account of the indeterminate quantity λ; the advantage of this expression is, that by means of it we can determine λ, and $\therefore - R$, the pressure. The equations of the equilibrium of a material point being independent, two or more of them may obtain without the others having place; this is an advantage connected with the resolution of the forces parallel to three rectangular coordinates.—*See* Notes to page 249.

CHAPTER II.

(*a*) Let v be the velocity common to all bodies on the earth's surface, and f the force with which a given body M is actuated in consequence of this velocity, and let the body be sollicited by any new force f', $a \; b \; c$ being the components of f resolved parallel to three rectangular axes, and $a' \; b' \; c'$ the components of f' resolved parallel to the same, by the notes to preceding chapter F, the resultant of $f, f' = \sqrt{(a+a')^2+(b+b')^2+(c+c')^2}$.

(*b*) If $v = f\phi(f)$, $v' = f'\phi(f')$, $V = F\phi(F)$; the *relative* velocity of the body resolved parallel to the axis

of $a = \frac{(a+a').V}{F} - \frac{av}{f} = (a+a')\ \phi\ (F) - a.\ \phi\ (f)$; but as f' is very small relatively to f, we have by neglecting indefinitely small quantities of the second and higher orders, $F = f + \frac{aa' + bb' + cc'}{f}$ and $\phi\ (F) = \phi\ (f). + \frac{aa'+bb'+cc'}{f}.\ \phi'\ (f)$ $\because$ by substituting, the relative velocity of the body parallel to $a = a'\ \phi f + \frac{a}{f}.(aa'+bb'+cc').\ \phi'\ (f)$, parallel to $b = b'\ \phi\ (f,) + \frac{b}{f}\ (aa' + bb' + cc').\phi'(f)$, parallel to $c = c'\ \phi\ (f) + \frac{c}{f}.\ (aa'+bb'+cc').\ \phi'(f)$; if the direction of the impressed motion coincided with a, then the preceding expressions would become $a'\ (\phi f + \frac{a^2}{f}.\ \phi'\ (f))$; $\frac{ab}{f}.\ a'\ \phi'\ (f)$; $\frac{ac}{f}.\ a'\ \phi'\ (f)$.

(*c*) If $\phi'\ (f)$ does not vanish, the body, in consequence of the impressed force a' will have a relative velocity perpendicular to the direction of a, if b and c do not vanish, *i. e.* if the direction of a does not coincide with that of the motion of the earth; but as in all cases, those perpendicular velocities vanish; it follows, that $\phi'\ (f)$ vanishes and therefore $\phi\ (f)$ is constant, consequently the function of the velocity which expresses the force is f.

(*d*) If $\phi\ (f)$ consisted of several terms, $\phi'(f)$ could never be = to cypher, if f was not = to cypher; if $\because v$ was not $\div l$ to f; $\phi\ (f)$ consists of several tems, and also the velocity of the earth must be such as to render $\phi'\ (f) = o$; which cannot be reconciled with the known fact, that the velocity of the earth is different at different seasons of the same year and at corresponding seasons of different years.

(*e*) Some philosophers hold that this discussion, as to the $\div$nality of the force to the velocity, is altogether superfluous, as we are not sure that forces such as we *conceive*

them, exist without our conceptions; for what is termed force is only an abstraction, which we make use of to enable us to subject the laws of motion to the calculus; the *true law of nature* is that discovered by Newton, namely, that the velocity communicated by the sun in an instant to the planets, is in the inverse ratio of the square of the distances, and all his physical discoveries might be deduced without using the term force instead of velocity; it follows from this law, that whatever is ∻l to the velocity follows necessarily the same ∻, so that if Newton assumed that the velocity $f \backsim v^2$, he would have obtained the same results, but then he should say, not that f but that $\sqrt{f}$ varied as $\frac{1}{d^2}$.

(f) If the spaces successively described in = times, constitute an increasing series, the motion of the body is said to be accelerated; if they constitute a decreasing series the motion is retarded; in these cases the measure of the velocity is obtained by determining the space which would be described in a given time, if all causes of acceleration or retardation were to cease after the point attains that position; now as the change in the velocity may be diminished indefinitely by diminishing the space, and ∵ the time in which it is described, if dv ds dt be the indefinitely small increments or decrements of v, s, t, &c. the spaces described in the times dt, immediately preceding and subsequent to the time in which the velocity is required to be estimated, are $(v \pm dv).\, dt$; but as one of those spaces is described with a greater and the other with a less velocity than that with which ds is described, we have $(v+dv).\, dt > ds > (v-dv).\, dt$; but when dv and ∵ dt are indefinitely diminished, the extreme quantities approach within any assignable difference, ∵ $v.\, dt$ and ds, which always exist between them, must differ by a quantity less than any assignable difference ∵ $v = \frac{ds}{dt}$, whatever be

the nature of the force; hence if on an assumed line = portions be taken representing the = intervals of time, and if at these points of equal section perpendiculars to the assumed line be drawn, representing the velocities acquired at the corresponding moments, the areas formed by connecting the extremities of the perpendiculars will represent the spaces, this area will be made up of a series of trapezia, if the velocity increases per saltum; if however the intervals of time be increased indefinitely, the velocity will continually approach to that in which the variation is continued, and the figure will be a nearer representation of the space actually described: its limit is a curvilinear area, on the base of which the elements of time are taken the ordinates being $\div$l to the velocities; this limit differs from the figure of which it is the limit, by a triangle under one of the equal subdivisions of the base, which are supposed to represent dt the element of time, and the difference between the extreme ordinates, hence when dt is indefinitely small this difference vanishes.

(g) If the velocity receives = increments in = times, *i. e.* if it be uniformly increased, the velocity is as the number of = increments, or as the number of = portions of time from the commencement of the motion, *i. e.* as the times, $\because$ in this case, if on the line representing the time, ordinates be erected, they will be as the corresponding abscissæ, the velocity being supposed = to o, when the time = o, and the locus of the extremities of these ordinates will be a right line diverging from the given line at the point where velocity and time = o, and the area of this triangle at the end of any time will represent the space described, and as the triangles representing the spaces described in the given intervals of time are always similar, the spaces described are as the squares of the times of their description, or of the last acquired velocities, $\because$ the spaces described in $1''$, $2''$, $3''$, &c. are as 1, 4, 9, &c. and

the spaces described in the 1st, 2nd, 3rd, &c. = moments are as the difference of the squares of these moments *i, e,* as 1 3 5 7 9, &c.

(*h*) If a body at the commencement is actuated by any finite velocity, then the space described is geometrically represented by a trapezium, one of whose sides is the initial velocity, and the other an ordinate, = to the sum of this ordinate and of the ordinate which would express the velocity of the body, had it fallen freely in the same time; if v' be the initial velocity, $v = v' \pm ft$ ∴ $vt = v't \pm \frac{ft^2}{2}$ ∴ s the space described $= v't \pm \frac{ft^2}{2}$; if the body moved with a uniform velocity v during t, s, the space described $= vt$. If it acquired the velocity v in the time t, by being urged by an uniform force from a state of rest, s' the space described would be $\frac{v.t}{2}$ ∴ $s : s' :: 2 : 1$.

(*i*) Let v t s represent the velocity, time, and space, and f the accelerating force $= \frac{v}{t}, = \frac{vt}{t^2} = \frac{v^2}{vt}$ *i, e,* $= \frac{2s}{t^2} = \frac{v^2}{2s}$, f denoting the unit of velocity or the velocity generated in a unit of time, ∴ $s = \frac{ft^2}{2}$; $v^2 = 2f.s$.

(*k*) The force acting parallel to the inclined plane being to the force of gravity which is constant, as h the height of the plane to l the ~~height~~ length, *i. e.* in a constant ratio, a body moving down an inclined plane has its motion uniformly accelerated, ∴ if v' s' represent the spaces described by a body descending down an inclined plane in any time t, and v' the acquired velocity, f' the accelerating force, we have $f' = \frac{h}{l}f$, $v' = \frac{h}{l}.ft$; $s' = \frac{h}{l}.\frac{ft^2}{2}$ $v'^2 = 2\,s'.\frac{h}{l}.f$; ∴ $v' : v :: h : l$; $s' : s :: h : l$; if $s' = l$ then we have $l = \frac{h}{l}.\frac{ft^2}{2}$ ∴ $t = l.\sqrt{\frac{2}{fh}}$; as $\frac{2h}{f}$. expresses the square of the time acquired in falling down the vertical, and as we have $t^2 =$

$\frac{2ls}{fh}$; when this is = the square of the time acquired in falling down the vertical, we have $ls = h^2$, $\therefore s = \frac{h^2}{l}$, $\because$ if a perpendicular be let fall from the right angle on the plane, it will cut off a portion of the plane, which will be described in the same time as the perpendicular height; and if a circle be described on this height as diameter, it is evident from what has been just established, that all chords drawn from *its* extremity to the circumference, are described in the same time as the diameter, $\because$ in = times.

(*l*) Let v' the velocity of projection be resolved into two, of which one is vertical and the other parallel to the horizon, and let e be the elevation of the line of direction, we have $v'.\ \sin. e$, $v'.\ \cos. e$, for the velocity of projection estimated in the direction of x and y respectively; $v'.\cos. e$ is the motion parallel to the horizon, $v'.\ \sin. e - ft$ is the vertical motion of the projectile, $\because$ if in the equations given in page 416, we make $v' = o$, we shall have for the height of ascent $s = \frac{v'^2.\ \sin. e}{2f}$, and $t = \frac{v'.\ \sin. e}{f}$, for the time of ascent, $\because$ $\frac{2v'.\ \sin. e}{f}$ for the time of flight; to find the horizontal range, the velocity $v'.\ \cos. e$, must be multiplied into $2t$ or its equivalent $2\frac{v'.\ \sin. e}{f}$, it $\because$ is equal to $\frac{v'^2.\ \sin. 2e}{f}$; therefore it is a maximum when $e = 45$, and for any elevations which are complements of each other, the horizontal ranges are =, the coordinates of the place of the body for any time t, are $x = v'.\ t \cos. e$, $y = v'.\ t \sin. e. - \frac{ft^2}{2}$, $\because$ as t is the same in these two equations we obtain by eliminating it and substituting $2fh$ for v'^2, $y = x \tan. e - \frac{x^2}{4h \cos.^2 e}$ whieh is the equation of a parabola, $4\ h \cos.^2 e$

is the principal parameter, and h the parameter of the diameter passing through the point of projection, hence being given of x y e h, any three, the fourth may be found.

(m) Let the arc described, reckoning from the lowest point $= s$, the ordinate $= y$, and the vertical abscissa $= x$, the origin of the coordinates being at the lowest point, if $b =$ the value of x at the commencement of the motion; v the velocity at the end of any time t, is the same as would be acquired by falling through the vertical height, $b - x$, *i. e.* $v = \sqrt{2g.(b-x)} = -\frac{ds}{dt}$; *see* Note($x$) $\because$ $dt = -\frac{ds}{\sqrt{2g(b-x)}}$, the negative sign being taken, because s diminishes according as t increases; but as $ds = \sqrt{dx^2 + dy^2}$, $y^2 = 2rx - x^2$. we obtain by substituting, $ds = \frac{rdx}{\sqrt{2rx - x^2}}$ $\because$ $dt = -\frac{rdx}{\sqrt{(2rx - x^2)2g(b-x)}}$, if the oscillations are very small, x may be neglected relatively to r, then the value of dt becomes $\frac{-r.\,dx}{\sqrt{2rx\,(2g(b-x)}}$, $= \frac{1}{2}.\sqrt{\frac{r}{g}} \times \frac{-dx}{\sqrt{bx - x^2}}$ the integral of the variable factor $=$ arc $\left(\cos. = \frac{2x-b}{b}\right) = \pi$, when we integrate from $x = b$ to $x = o$, $\because$ the time of a semioscillation $= \frac{1}{2}\pi.\sqrt{\frac{r}{g}}$;

(n) Hence it follows, that provided the amplitudes be inconsiderable, the time of oscillation is always the *same*, when r and g are given, when these quantities vary the time varies as $\sqrt{\frac{r}{g}}$ *i. e.* directly as the square roots of the lengths of the pendulums, and inversely as the square root of the force of gravity.—*See* Note (s) page 355. As 3π, 5π, &c. and in general any odd multiple of π satisfies the pre-

ceding integral, of $\frac{r\,dx}{\sqrt{bx-bx^2}}$; it is evident that the body arrives at the lowest point an indefinite number of times, which are separated from each other by the time $\pi\sqrt{\frac{r}{g}}$; hence it follows, that if all obstacles were removed, the number of oscillations would be infinite and the time of each =.

The value of dt may be made to assume the form $\frac{1}{2}\sqrt{\frac{r}{g}}.\frac{-dx}{\sqrt{bx-x^2}}\;\frac{1}{\sqrt{\left(1-\frac{x}{2r}\right)}}$ = (by developing the factor

$$\left(1-\frac{x}{2r}\right)^{-\frac{1}{2}} \text{ in a series) } \frac{1}{2}\sqrt{\frac{r}{g}}.\;\frac{-dx}{\sqrt{bx-x^2}}.$$

$\left(1+\frac{1}{2}.\frac{x}{2r}+\frac{1.3}{2.4}.\frac{x^2}{4r^2}+\text{\&c.}\right)$, if $\frac{dx}{\sqrt{bx-x^2}}$ be multiplied by each term of this series the resulting terms will be of the form $\frac{-x^m dx}{\sqrt{bx-x^2}}$, of which the integral when taken between the limits $x=o$, $x=b$, is

$\frac{(\frac{1}{2}b)^m.\ \pi.\ 1.\ 3,\ 5,\ \text{\&c.}\ (2m-3).\ (2m-1)}{1.\ 2.\ 3\ldots m}$ from which if m be made successively = 0 1, 2, &c. the value of t becomes $T=\frac{1}{2}\pi.\sqrt{\frac{r}{g}}.\left(1+(\frac{1}{2})^2.\frac{b}{2.\ r}+\ldots\left(\frac{1.3(\ldots)(2m-1)}{2.\ 4-\ \ 2m}\right)^2.\right.$

$\left.\frac{b^m.}{2^m.\ r^m}+\text{\&c.}\right)$; b is the versed sine of the arc described, which when it is inconsiderable may evidently be neglected, in this case the value of t is the same as was obtained in the preceding page; when great accuracy is required, the two first terms of the series are retained, in that case the aberration from isochronism varies as the square of the sine of half of the amplitude of the arc described.

(*o*) As τ the time of fall in the vertical through a space $=$ to half the length of the pendulum $= \sqrt{\frac{r}{g}}$, we have $T : \tau :: \pi.\sqrt{\frac{r}{g}} : \sqrt{\frac{r}{g}} :: \pi : 1$; if t' be the time employed to describe the chord of the indefinitely small arc, as this time $=$ the time of falling vertically through the diameter $2r$, *see* preceding Note $\because$ it is $= \sqrt{\frac{4r}{g}}$, $\because \frac{T}{2} : t' :: \frac{\pi}{2}.\sqrt{\frac{r}{g}} : \sqrt{\frac{4r}{g}}$, *i. e.* $\pi : 4$ or as the periphery of a circle to four times the diameter; hence it is evident that the chord is not the line of swiftest descent, *see* Note (*p*).

Naming $\tilde{\omega} =$ the angular velocity, we have $v = r.\tilde{\omega}$: $\because$ $\tilde{\omega} = \frac{v}{r} = \frac{\sqrt{2g.(b-x)}}{r}$ but if a be the angular distance from the vertical at the commencement of the motion, and θ the angular distance at the end of any time t, we have $b = r.\cos. a$, $x = r.\cos.\theta$, $\because \tilde{\omega} = \sqrt{\left(\frac{2g}{r}\right.(\cos.\theta - \cos. a)}$. The acccelerating force in any point, $=$ the force of gravity resolved in the direction of the tangent; $\because$ if any vertical line be assumed to represent the force of gravity, the accelerating or tangential force $=$ this line multiplied into the sine of the angular distance from the lowest point. If the body, instead of falling freely, had a velocity at the commencement of the motion due to the height h, then the velocity at any point of which the height $= x$, is $\sqrt{2g(h+b-x)}$ and $= o$, when $x = h + b$, $\because$ when the body attains a height $= h + b$, it ceases to rise; v will never vanish when $h + b$ is $>$ than the diameter which is the greatest value of x, $\because$ the body will gyrate for ever with a variable velocity, the greatest being when at the lowest, and the least at the highest extremity of the vertical diameter. When a body M attached to a string describes an arc of a curve, the

tension at the point to which the string is attached, arises from the centrifugal force and the force of gravity resolved in the direction of the string; if the arc described be the arc of a circle, the part of the force of gravity which acts in the direction of the string $= g.\left(\frac{r-x}{r}\right)$, r being the length of the string, and x the distance above the lowest point; the centrifugal force $= \frac{v^2}{r} = 2g.\left(\frac{b-x}{r}\right)$; this always acts from the centre; $\because$ the whole tension $= Mg. \left(\frac{r+2b-3x}{r}\right)$; if M falls from an horizontal diameter, $r=b$, and the tension at any point $= 3Mg.\left(\frac{r-x}{r}\right)$, *i. e.* three times the effect of the weight resolved in the direction of the radius vector. If the pendulum fell from the vertical position freely, then $b=2r$ and $\because$ the tension $= Mg\left(\frac{5r-3x}{r}\right)$, and when $x = o$, it is equal $5Mg.$ or five times the weight; making $Mg\left(\frac{r+2b-3x}{r}\right) = Mg$ we obtain $x=\frac{2}{3}b$, the value of x when the tension $=$ the weight; when $x = \frac{r+2b}{3}$ the tension $= o$; but as x can never exceed either b or $2r$; when it is respectively $=$ these quantities, we have $b=r$, $b=\frac{5r}{2}$, if $b \angle r$ then the force of gravity resolved in the direction of the string is directed from the centre, $\therefore$ this point then suffers a tension from both causes; if $b > \frac{5r}{2}$, the centrifugal force is throughout $>$ than weight, $\because$ the whole tension can never vanish, but if b is not $\angle r$ or $> \frac{5r}{2}$ the tension may vanish; at this point the body will quit the circle, and as its direction will be that of a tangent to this circle it will describe a parabola. In a cycloid if

a body falls freely from the extremity of the base, the pressure arising from the weight resolved in the direction of the string $= g.\frac{\sqrt{a-x}}{\sqrt{a}}$, and likewise that produced by the centrifugal force $= \frac{2g(a-x)}{2.\sqrt{a(a-x)}} = g.\frac{\sqrt{a-x}}{\sqrt{a}}$, hence at the lowest point, the entire tension = twice the weight; in any other point the entire tension is to weight, as twice the cosine of the inclination of the tangent to the horizon to radius; hence, when the body falls from the horizontal base, they are equal at the point of the cycloidal arc where the tangent is inclined at an angle of 60° to the horizon.

(q) Calling this space x, we have $2x :: r :: \pi^2.\frac{r}{g} : \frac{r}{g} :: \pi^2 : 1$; the equation $T = \pi.\sqrt{\frac{r}{g}}$ gives likewise a very exact measure of g, for if l be the length of this pendulum vibrating seconds, we get $g = \pi^2.l$, which expresses the velocity generated in one second by the space which would be described with that velocity continued uniformly for that time, the space described by a body falling from rest in a second is one half of this, or $\pi^2.\frac{l}{2}$; substituting for π, l their numerical values given in the text we obtain $3^m,66107$ for the space described in the first second.

As the sine of the angle which the tangent at any point of a vertical curve makes with the horizon, $= \frac{dx}{ds}$, the accelerating force along the tangent $= g.\frac{dx}{ds} = \frac{g}{2a}.s$; (when the curve described is a cycloid, in consequence of the equation of the cycloid $s^2=4ax$), the preceding is the expression for the accelerating force, in any curve whatever which renders it tautochronous, ∵ this force is at each instant ∻l to the length of the arc to be described, in order

to arrive at the lowest point of the curve; and conversely if $\frac{dx}{ds} = As$, it is easy to shew that when the curve is one of *single curvature existing in a vertical plane*, its equation is that of a cycloid, for by integrating the preceding equation, and then eliminating s between the integral $x = \frac{1}{2} As^2$, and $\frac{dx}{ds} = As$, we obtain $\frac{1}{2A} \cdot \frac{dx^2}{x} = ds^2 = \overline{dy^2 + dx^{2\prime} + dz^2}$; $\therefore \frac{1}{A} x = s^2$, if the curve is one of single curvature inclined to the horizon at an angle $= \theta$, then if y' x' be the coordinates in that plane, we have $y' = y$, $x = x'.\ \sin.\ \theta$, consequently the equation of the curve is $\frac{1}{2A} . x'.\ \sin.\ \theta = s^2$; note the relation $\frac{1}{A} \frac{dx^2}{x} = ds^2 =$ generally $dx^2 + dy^2 + dz^2$, and as $\frac{1}{2A} .\ x = s^2$ is independent of z y, these quantities may vary according to any law whatever, which satisfies the equation $ds^2 = dx^2 + dy^2 + dz^2$; $\because$ any curve of double curvature which arises from wrapping a cycloid around a vertical cylinder of which the base is a continuous curve, will satisfy the preceding conditions, and $\because$ be tauto chronous; and conversely such a curve so unfolded as that it might entirely exist in the same plane would continue to possess this property, and $\because$ from what has been stated above, would necessarily be a cycloid. We might investigate a priori, the time necessary for a body to describe any portion of a cycloidal arc on the hypothesis, that it moves with an initial velocity represented by $\sqrt{2gh}$, for let h' represent the vertical ordinate at the commencement of the motion, the origin being as before at the lowest point, and x the ordinate after any time t, we have $v^2 = 2g(h + h' - x)$; $\because dt = \frac{-ds}{\sqrt{2g(h + h' - x)}}$ but as $ds = dx.\sqrt{\frac{a}{x}}$ by substituting we have $dt = -\sqrt{\frac{a}{2g}} . \frac{dx}{(x(h + h' - x))^{\frac{1}{2}}}$ $\because$ integrating we obtain $t =$

$$\left(\frac{a}{2g}\right)^{\frac{1}{2}} \text{arc}\left(\cos. = \frac{\frac{x-(h+h')}{2}}{\frac{(h+h')}{2}}\right. + C; \text{ as } t=o \text{ when } x=h',$$

$$C=-\left(\frac{a}{2g}\right)^{\frac{1}{2}}.\ \text{arc}\left(\cos. = \frac{h'-h}{h'+h}\right);$$ and when $x=o$, *i e*, at the lowest point $t=\left(\frac{a}{2g}\right)^{\frac{1}{2}}.\ \left(\pi-\text{arc}\left(\cos. = \frac{h'-h}{h+h}\right)\right)$; if $h=o$ *i.e.* if the initial velocity vanishes $t=\pi.\left(\frac{a}{2g}\right)^{\frac{1}{2}}$, ∵ as h does not occur in this expression, the time is independent of the amplitude of the arc described; it appears from a comparison of this value of t with that given in page 418, that the oscillations in a cycloid are isochronous with the indefinitely small vibrations in a circle, of which the radius is equal to twice the axis of the cycloid.

Huygen's contrivance depended on the known property of cyloids, namely, that their evolute was a curve = and similar to the given cycloid, hence it follows, that if two metallic curves, each consisting of an inverted semi cycloid with an horizontal base touched at their upper extremities; and if at their point of contact, the thread of the pendulum was attached (its length being equal to either of the semi cycloids,) when it is enveloped on the curves, its other extremity will trace a curve = and similar to the given curve, having its axis however in an opposite direction.

(r) From the times of vibration and lengths of these pendulums being the same, the times of falling down the = axes are the same, ∵ all bodies falling freely are equally accelerated by the force of gravity.

It is easy to shew that the time of describing the chord of a semi cycloidal arc is to the time of describing the arc, as the chord to half the base of the cycloid, which is evidently a ratio of major inequality.—*See* Note (p), page 425.

(p) In investigating the nature of the curve of swiftest descent in a *vacuo*, it is easy to shew that if the entire line be supposed to be described in the shortest possible time, so any portion of this line intercepted between two assumed points is described in a less time than any other curve joining these two points; hence if $x\,y$ be the vertical and horizontal coordinates of any point, reckoning from the point whence the body has commenced to move, s the corresponding arc of the curve, the time of describing $ds = \frac{ds}{\sqrt{2gx}}$, in like manner if a point indefinitely near to the first point be taken whose coordinates are $x'y'$ and the corresponding arc described from commencement $= s'$, we have $x' = x + dx$, $s' = s + ds$, and the time of describing ds' $= \frac{ds'}{\sqrt{2gx'}}$, $\because$ the time of describing the entire arc made up of $ds' + ds = \frac{ds}{\sqrt{2gx}} + \frac{ds'}{\sqrt{2gx'}}$, therefore we have $o = \delta\left(\frac{ds}{\sqrt{2gx}} + \frac{ds'}{\sqrt{2gx'}}\right)$, but from the conditions of the problem $x\,x'$ are independent of these variations $\therefore \delta x, \delta x' = o$, and consequently $\frac{\delta.\,ds}{\sqrt{x.}} + \frac{\delta.ds'}{\sqrt{x'}} = o$; and as $dx\,dx'$ have no variations, $\delta.\,ds = \delta.\,d\,\sqrt{dy^2 + dx^2} = \frac{dy.\ \delta\,dy}{ds}$, &c. $\because$ by substituting we have $\frac{dy.\ \delta\,dy}{ds\,\sqrt{x}} + \frac{dy'.\ \delta\,dy'}{ds'\,\sqrt{x'}} = o$, but $dy + dy'$ is constant, therefore $\delta\,dy = -\,\delta\,dy'$, consequently $\frac{dy}{ds.\sqrt{x}} - \frac{dy'}{ds.\sqrt{x'}} = o$, *i. e.* $d.\left(\frac{dy}{ds\,\sqrt{x}}\right) = o$, (for the two points $x\,y$, $x'\,y'$, are continuous,) and $\frac{dy}{ds.\sqrt{x}} = \mathrm{C}$; now as $\frac{dy}{ds}$ is the sine of the angle which the tangent makes with

the axis of x, when the arc is horizontal, this angle is right, and $\because \frac{1}{\sqrt{a}} = C$, ($a$ being the value of y at this point,) $\because \frac{dy}{ds} = \sqrt{\frac{x}{a}}$ by squaring and substituting we get $dy^2 . \left(1 - \frac{x}{a}\right) = dx^2 \frac{x}{a} \because dy = dx . \frac{x^{\frac{1}{2}}}{\sqrt{a-x}}$ and $ds = dx \sqrt{\frac{a}{a-x}} \because s = -2\sqrt{a(a-x)} + C'$; but when $s = o$, $x = o$, $\because C' = 2a$, and $s = 2a - 2\sqrt{a(a-x)}$, which is the equation of a cycloid, of which the axis is a, the arc being measured from the horizontal base.

If the curve is not required to pass between two given points, but between two given curves, then it would not be difficult to shew that the required curve is a cycloid meeting the two given curves at right angles.

(s) In an indefinitely small portion of time, the quantity by which the body is deflected from the tangent to the circle, which measures the centripetal and consequently the centrifugal force, is the versed sine of the arc described; and as this is the space which the central force causes a body to describe, the force of gravity will be to the centrifugal force as the space described, in consequence of the action of gravity in this time, to this versed sine.

(t) Calling f the accelerating force, we have $f = \frac{2\,dr}{dt^2}$; $dr = \frac{ds^2}{2.r}, \because f = \frac{ds^2}{dt^2 r}$ but $\frac{ds}{dt} = v \because f = \frac{v^2}{r}$; the curve described being a circle in which the deflection from the tangent is always the same, the force acting on the point is a constant accelerating force; hence as v^2 always $= 2gh$, we have $\frac{v^2}{r} = f = \frac{2gh}{r}$ and $\frac{f}{g} = \frac{2h}{r}$ which gives generally the relation between the centrifugal force in a circle and the force of gravity, and they are $=$ when $h = \frac{r}{2}$; *i. e.* the body must fall through half the radius in order

to acquire the velocity which renders the centrifugal force equal to the gravity; if $P =$ the time of revolution we have $v = \frac{2\pi r}{P} \because f = \frac{4\pi^2 r}{P^2}$, this expression gives $\frac{11154}{32 \cdot 12530} = \frac{1}{288}$ for the ratio of the centrifugal force to the force of gravity at the equator, and because when r is given, f varies inversely as P^2, if P' be the time of the earth's rotation when the centrifugal force $=$ the force of gravity, we have $P^2 : P'^2 :: 289 : 1$ therefore $P' = \frac{P}{17}$, hence if the earth revolved on its axis in the 17th part of a day, *i, e*, in $1^h, 24' \ 28\frac{1}{2}''$ the centrifugal force would be equal to the gravity. *See* Notes to Chapter VIII. Vol. II

It follows from the expression $f = \frac{4\pi^2 r}{P^2}$, that the centrifugal force on the earth's surface is greatest at the equator, and that it decreases as the cosine of latitude; however as its direction is inclined to the direction of gravity it is not entirely efficacious at any *parallel*, and by a resolution of forces it may be shewn that the efficacious part is to the whole centrifugal force at the parallel, as the cosine of the latitude λ to the radius, and therefore to the centrifugal force at the equator as $\cos. \lambda^2 : 1$; the part of the resolved force which acts perpendicularly to the direction of gravity, and is therefore inefficacious, varies as $\sin. \lambda. \cos. \lambda$.

(*u*) The force which is in equilibrio with the centrifugal force is $\because$ the measure of the pressure arising from the tendency of the body to recede in the direction of the tangent; hence, by note (*t*) it is $\frac{2dr}{dt^2} = \frac{v^2}{r}$; (*r being the radius of curvature*,) the effect of the part of the force resolved in the direction of dr is therefore to produce a continued change in the direction of the motion; and the effect of the other part is evidently to *accelerate* or *retard* the motion of the body, its variation $= \frac{1}{n^2 . r} . \sqrt{1 - \frac{p^2}{\rho^2}}$; ρ being the radius vector.

(v) Calling $d\rho$ the part of the radius vector intercepted betwen the curve and the tangent, ds the arc and c the chord of curvature, we have $f=\frac{2d\rho}{dt^2}$, but $d\rho=\frac{ds^2}{c}$ $\therefore$ $f=\frac{ds^2}{dt^2.\,c}=\frac{v^2}{c}$, this expression is general, and true independently of the equal description of areas; on the hypothesis that the areas are $\div\!\!\!\div$ 1 to the times, $v \backsim \frac{1}{p}$, p being a perpendicular let fall from the centre of force on tangent, and $\therefore$ $f \backsim \frac{1}{p^2.\,c}$ which is one of Newton's expressions.

Let x be the space through which the body should fall to acquire the velocity in the curve, the velocity acquired in falling through dc is to the velocity with which the arc is described, as $2dc : ds$; and $dc : x ::$ as the square of the velocity acquired in falling through dc to the square of the velocity with which dc is described, $\therefore dc : x :: 4dc^2 : ds^2$ $\therefore x=\frac{ds^2}{4dc}=\frac{c}{4}$, *i. e.* a body falls through one-fourth of the chord of curvature to acquire the velocity in the curve.

(v) It is by taking the function of the radius vector, which is equal to this limit, that Newton determines the expression for force in conic section, spiral, &c., *see* Princip. Math. sec. 2 and 3. It would not be difficult to shew by reasoning precisely similar to that in pages 249, 250, that if a body is attracted to two fixed points which are not in the same plane as that in which it moves, the body will describe = solids in equal times about the line connecting the attracting points.

The proposition established in page 246 may be thus proved, by what is stated in page 249, $X=\frac{d^2x}{dt^2}$, $Y=\frac{d^2.y}{dt^2}$ $Z=\frac{d^2z}{dt^2}$; multiplying the first equation by y and z, the second by x and z, and the third by x and y, we obtain by subtracting, $\frac{d^2y}{dt^2}.x-\frac{d^2x}{dt^2}.y=Y.x-X.y$, $\frac{d^2y}{dt^2}.z-\frac{d^2z}{dt^2}.y$

$= \text{Y}.z - \text{Z}.y,\ \frac{d^2z}{dt^2}.x - \frac{d^2x}{dt^2}.z = \text{Z}.x - \text{X}z$, by integrating we obtain $\frac{dy.x - dx.y}{dt} = \text{C} + \int (\text{Y}x - \text{X}y)\,dt : \frac{dy}{dt}.z - \frac{dz}{dt}.y = \text{C}' + \int (\text{Y}z - \text{Z}y)\,dt ; \frac{dzx}{dt}. - \frac{dxz}{dt} = \text{C}'' + \int (\text{Z}x - \text{X}z)dt$; but when the force is directed to a fixed point, which is the origin of $x\ y\ z$, $(\text{Y}x - \text{X}y)$, $(\text{Y}z - \text{Z}y)$, $(\text{Z}x - \text{X}z)$, are respectively $= o$, *see* Chapter IV. Note (h), $\therefore dyx - dxy = \text{C}.dt$, a constant quantity, but this quantity is evidently $=$ to the projection of the element of the area on the plane $x\ y$, for let ρ be the projection of the radius vector, ψ the angle which it makes with x and y, we have $x = \rho \cos.\psi, y = \rho \sin.\psi$, $\therefore xdy - ydx = \rho^2.\,d\psi$, which is the element of the area. The quantities C C′ C″ depend on the nature of the curve described. In the case of a conic section, origin being in the focus, they are respectively ∻ to the cosines of the inclinations of the planes xy, xz, yz, to the plane in which the body moves, multiplied by the square root of the parameter.

Multiplying each of the preceding equations by the variable which does not occur in it, and then adding them together we obtain the equation $o = \text{C}z + \text{C}'y + \text{C}''x$, which shews that when a body is acted on by a force directed to a fixed point, it will describe a curve of single curvature.

(x) By referring the position of a point in space to rectangular coordinates, every species of curvilinear motion may be reduced to *two* or *three* rectilinear motions, according as the curve described is of single or double curvature, for the position of a point in space is completely determined when we can determine the position of its projections on three rectangular axes, each coordinate is the rectilinear space described by the point parallel to the axis to which it is referred, it will $\therefore$ be some given function of the time; if we could determine these functions for the *three* coordinates, the species of the curve described would be given, by eliminating the time by means of the three equations be-

tween the coordinates and the time. The space s being considered a function of the time t it is easy to shew that the velocity is $=\frac{ds}{dt}$, and f the force is $\div 1$ to $\frac{d^2s}{dt^2}$, for t receiving the increment dt, then $s = \phi\ (t)$ becomes $s' = \phi(t+dt)$ and $s'-s = \frac{ds}{dt}.\ dt + \frac{d^2s}{dt^2}\ dt^2 + \frac{d^3s}{dt^3}.\ dt^3 +$ &c.; if dt be considered as indefinitely small, in which case we can consider the velocity as uniform and the force as constant, $\frac{ds}{dt}$ being the coefficient of dt expresses the velocity, and $\frac{d^2s}{dt^2}$ being the coefficient of dt^2, it is $\div 1$ to the force; $\because$ if the action of the forces soliciting the point should cease suddenly $\frac{d^2s}{dt^2}$ would vanish, and the point would move with an uniform velocity, if instead of vanishing $\frac{d^2s}{dt^2}$ became constant, then $\frac{d^3s}{dt^3}$ and all subsequent coefficients would vanish, and the motion of the point would be composed of a uniform motion and of a motion uniformly accelerated, both commencing at the same instant; now if f represents the force, it is evident that $f.\ dt = dv, = d.\frac{ds}{dt} = \frac{d^2s}{dt.}$

(y) Let P Q R represent the resultants of all the forces which act on the point parallel to $x\ y\ z$ respectively, we have $\frac{d^2x}{dt^2} = \text{P}, \frac{d^2y}{dt^2} = \text{Q}, \frac{d^2z}{dt^2} = \text{R}$, consequently if the point was actuated by the forces

$$-\ d.\frac{dx}{dt} + \text{P};\ -\ d.\frac{dy}{dt} + \text{Q};\ -\ \frac{dz.}{dt.} + \text{R}$$

they would keep it in an equilibrium; hence from what has been already established in Notes, page 354, we have

$$\left(d.\frac{dx}{dt}-\mathrm{P}\right).\ \delta x+\left(d.\frac{dy}{dt}-\mathrm{Q}\right).\ \delta y+\left(d.\frac{dz}{dt}-\mathrm{R}\right).\ \delta z=o;$$

if the point be free we shall have, as is stated in the text, the coefficients of δx, δy, δz, separately $=o$; *i. e.* $\frac{d^2x}{dt^2}=\mathrm{P}$; $\frac{d^2y}{dt^2}=\mathrm{Q}$, $\frac{d^2z}{dt^2}=\mathrm{R}$; but if the point is constrained to move on a curve or surface, by means of the equations to this curve or surface, we can eliminate as many of the variations $\delta x\ \delta y\ \delta z$ as there are equations; the coefficients of the remainder may be put $=$ to cypher; it appears from this process, which is that made use of by Laplace in his Celestial Mechanics, how the laws of the motion of a point may be deduced from those of their equilibrium: we shall see in the sixth chapter that the laws of the motion of any system of bodies may be reduced to those of their equilibrium; if P Q R are given in functions of the coordinates, then by integrating twice we obtain $x\ y\ z$ in a function of the time; two constant arbitrary quantities are introduced by these integrations; the first depends on the velocity of the point at a given instant, the second depends on the position of the point at the same instant: if $x\ y\ z$ came out respectively $=a.f(t)$, $b.f(t)$, $c.f(t)$, the point will move in a right line, the cosines of the angles which it makes with $x\ y\ z=$

$$\frac{a}{\sqrt{a^2+b^2+c^2}}\quad\frac{b}{\sqrt{a^2+b^2+c^2}},\quad\frac{c}{\sqrt{a^2+b^2+c^2}};$$

the constant quantities $a\ b\ c$ depend on the nature of the function $f(t)$, if $f(t)=t$ then $a\ b\ c$ represent the uniform velocities parallel to $x\ y\ z$, and the uniform velocity of the point $=\sqrt{a^2+b^2+c^2}$; if $f(t)=t^2$; $a\ b\ c$ are proportional to the accelerating forces parallel to $a\ b\ c$, and the point will move with an uniformly accelerated motion represented by $\sqrt{a^2+b^2+c^2}$; if $x=a'.f(t)+b'\,\mathrm{F}(t)$; $y=c\,f(t)+d.\,\mathrm{F}(t)$, $z=e\,f(t)+g.\,\mathrm{F}(t)$, the path of the point will be a curve, however it will be of single curva-

tion; for by eliminating t we obtain an equation of the form $Ax + By + Cz = o$, which is that of a plane; the simplest case of this form is $x = a't + b't^2$, $y = c't + d't^2$, $z = e't + g't^2$; eliminating t between the two first equations we shall obtain an equation of the second order between x and y, which is evidently a parabola from the relation which exists between the coefficients of the three first terms. If $x = f(t)$, $y = F(t)$ $z = \phi(t)$, all the points in the curve will not exist in the same plane. The law of the force being given, the investigation of the curve which this force causes to be described, is more difficult than the reverse problem of determining the force, velocity, &c. the nature of the curve being given, as the integrations which are required in the first case are much more difficult than the differentiations which determine the force and velocity in the second. It may be remarked here, that the number of the equations of condition of the motion of a material point is necessarily *less than three*; for if there were three equations of condition between the coordinates $x\ y\ z$, it is evident that if these equations were independent of the time, their resolution would give particular values for each of the coordinates, $\therefore$ the point could not move; and if the equations contained the time the values of $x\ y\ z$ are given in a function of the time, so that the motion of the point being determined *a priori* by the equations of condition, it cannot be modified by any accelerating force; if there were more than three equations of condition their simultaneous existence would imply a contradiction.

(z) As $\delta x\ \delta y\ \delta z$ are arbitrary they may be assumed $=$ to $dx\ dy\ dz$ respectively, in which case we have

$$d.\frac{dx}{dt}\,dx + d.\frac{dy}{dt}\,dy + d.\frac{dz}{dt}dz = P\,dx + Q\,dy + R\,dz;$$

$\therefore$ by integrating $\dfrac{dx^2 + dy^2 + dz^2}{dt^2} = C + 2\int(Pdx + Qdy + Rdz)$; if this integral $= f(x\,y\,z)$, then $v^2 = C + f(x\,y\,z)$

let A be the velocity corresponding to the coordinates $a\ b\ c$, then $A^2 = C + 2f(a\ b\ c)$, $\therefore v^2 - A^2 = 2f(x\ y\ z) - 2f(a\ b\ c)$, *i. e.* the difference of the squares of the velocities depends on the coordinates of the extreme points of the line described, $\therefore$ is independent of the line described; so that when the point describes a curve, the pressure of the moving point on the curve does not affect the velocity. The constant quantity C depends on the values of v and of $x\ y\ z$ at any given instant; when the moving point describes a curve returning into itself, the velocity is always the same at the same point, and if the velocities of two points of which one describes a curve while the other describes a right line, are equal at $=$ distances from the centre of force at any given instant, they will be equal at all other $=$ distances; if the force varies as the n^{th} power of s the distance from the centre, then $f(x\,y\,z) = s^{n+1}$, $\therefore v^2 - A^2 = s^{n+1} - a^{n+1}$, and $2dv.\ v = (n+1).\ s^n\,ds$, $\therefore$ by erecting in the line drawn from the centre ordinates $\doteqdot 1$ to s^n, the resulting figure will represent the square of the velocity, when n is positive, this figure is of the parabolic species, when it is negative it will be of the hyperbolic species; if $Pdx + Qdy + Rdz$ be an exact differential, then $\frac{dP}{dy} = \frac{dQ}{dx}$, $\frac{dP}{dz} = \frac{dR}{dx}$ &c. and P Q R must be functions of $x\ y\ z$ independently of the time; now if the centres to which the forces were directed had a motion in space, the time would be involved, and $\therefore Pdx + Qdy + Rdx$ would not be an exact differential; if P Q R arose from friction or the resistance of a fluid, the equation $Pdx + Qdy + Rdz$ would not satisfy the preceding conditions of integrability, for as in such cases P Q R depend on the velocities $\frac{dx}{dt}\frac{dy}{dt}\frac{dz}{dt}$, $Pdx + Qdy + Rdz$ cannot be an exact differential of $x\ y\ z$ considered as independent variables, consequently in order to integrate, we should in the expression $Pdx + Qdy + Rdz$ substitute for these variables and their differentials, their values

P P

in a function of the time, which supposes that the problem is already solved, $\because$ when the point to which the force is directed is in motion, or when the force arises from friction or resistance, the velocity involves the time and $Pdx + Qdy + Rdz$ is not an exact differential. When a point moves in a right line, the velocity is = to the element of the space $\div$ded by the element of the time, *i. e.* $v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt}$ this is also true for curvilinear motion, for if P Q R should suddenly cease, the velocity in the direction of each coordinate is uniform and $= \frac{dx}{dt}\ \frac{dy}{dt}\ \frac{dz}{dt}$ respectively; therefore v the velocity of the point will be uniform and its direction rectilinear, *i. e.* $v = \frac{ds}{dt} = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt}$; the rectilinear direction is that of the tangent, for if A, B, C, denote the angles which this direction makes with $x\ y\ z$, we have

$v.$ cos. A $= \frac{ds}{dt}.$ cos. A $= \frac{dx}{dt}$; and v cos. B $= \frac{ds}{dt}.$ cos. B $= \frac{dy}{dt}$, $v.$ cos. C $= \frac{ds}{dt}.$cos. C $= \frac{dz}{dt}$ $\because$ cos. A $= \frac{dx}{ds}$, cos. B $= \frac{dy}{ds}$, cos. C $= \frac{dz}{ds}$ which are the expressions for the angles, which any tangent makes with the coordinates, $\because$ the tangent coincides with the line along which the point would move if P Q R should suddenly cease.

If the point moves on any curve whatever, the centrifugal force $= \frac{v^2}{r}$, *see* Notes, page 428, and as this force acts in the direction of a normal to the curve, if all the accelerating forces which act on the point be resolved to two, of which one acts perpendicularly to the trajectory, and the other in the direction of the tangent, the resul-

tant of the first of these forces and of $\frac{v^2}{r}$, is the entire pressure of the point on the curve, and the resistance of the curve is an accelerating force = and contrary to this resultant, denoting the normal force by L, if A′, B′, C′, be the angles which it makes with $x\ y\ z$ respectively; by the Notes to page 431, we have $\frac{d^2x}{dt^2} = P + L \cos. A'$, $\frac{d^2y}{dt^2} = Q + L \cos. B'^2$, $\frac{d^2z}{dt^2} = R + L \cos. C'$; but since the normal is perpendicular to the tangent we have $\frac{dx}{ds} \cos. A' + \frac{dy}{ds} \cos. B' + \frac{dz}{ds}. \cos. C' = o$, we have also $\cos.^2 A' + \cos.^2 B' + \cos.^2 C' = 1$, ∵ between these five equations we can eliminate A′ B′ C′ L, and the resulting equation, which as of the second order being combined with the equations of the trajectory, which are given in each particular case, will determine the coordinates $x\ y\ z$ in a function of the time; if the three preceding equations be multiplied by $dx\ dy\ dz$ respectively, and then added together, we obtain

$$\frac{d^2x\ dx + d^2y\ dy + d^2z\ dz}{dt^2} = Pdx + Qdy + Rdz + L\ (\cos. A'\ dx + \cos. B'.\ dy + \cos. C'.\ dz)$$

as the latter part of the second member $= o$, we have, by substituting for the first member its value, $\frac{(d^2s.\ ds)}{dt^2} = P.dx + Q.dy + R.dz$; ∵ $\frac{d^2s}{dt^2} = P.\frac{dx}{ds} + Q.\frac{dy}{ds} + R.\frac{dz}{ds}$ *i. e.* the accelerating force resolved in the direction of the tangent, is equal to the second differential coefficient of the arc considered as a function of the time, which is an extension of what has been established in Notes, page 430; it likewise appears that the force in the direction of the tangent is totally independent of L; and also that when there is no

accelerating force, $\frac{d^2 s}{dt^2} = o$. It appears from what has been just established, that when the equations of condition of the motion of the material point are independent of the time, the resultant of the forces which are equivalent to the equations of condition is normal to the curve described by the point, for in that case $P'dx + Q'dy + R'dz = o$; P' Q' R' being the resultant of these forces resolved parallel to x y z respectively; but if these equations are functions of the time $P'dx + Q'dy + R'dz$ is not $= o$. If V denotes the resultant of all the accelerating forces which act on the point, and the θ angle which this resultant makes with the normal, V cos. θ expresses the resultant resolved in the direction of the normal, and when the curve described is of *single* curvature, $\frac{v^2}{r} +$ V cos. θ. expresses the entire pressure $=$ L; $= \frac{v^2}{r} +$ P. $\frac{dy}{ds}$ + Q. $\frac{dx}{ds}$, ∴ if the equation of the trajectory be given, and also the values of P, Q, in terms of x y, we can determine v, and ∴ L, and substituting for L this value, in the expressions for $\frac{d^2 x}{dt^2}$, $\frac{d^2 y}{dt^2}$ &c. we might by integrating, determine the position of the point at any given moment, and also its velocity. As the coordinates are arbitrary, if we make one of them to coincide with the normal to the curve, denoting by A', B', the angles which the radius of the osculating circle makes with the normal and with the coordinate, which is in the plane of the tangent, and by m, n, l, the angles which V, the resultant of all the forces, makes with the three coordinates, the force expressed by $\frac{v^2}{r}$ resolved parallel to these coordinates $= \frac{v^2}{r}$. cos. A, $\frac{v^2}{r}$. cos. B, $\frac{v^2}{r}$. cos. 90°; and V resolved parallel to these coordinates $=$ V cos. m, V cos. n, V cos. l, and as A m denote the inclination of the radius of curva-

ture, and of V to the normal, $\frac{v^2}{r}$. cos. A + V cos. m expresses the pressure of the point on the surface, V cos. n $+\frac{v^2}{r}$. cos. 90° expresses the force by which the point is moved; ∴ V cos.l + $\frac{v^2}{r}$. cos. B = the motion perpendicular to the tangent $=o$; hence, if V l v and r were given, we might determine B and ∴ the inclination of the plane of the osculating circle to the tangent plane, and when there is no accelerating force, $\frac{v^2}{r}$. cos. B $= o$, *i. e.* B=90, or the plane of the osculating circle is at right angles to the surface $\frac{v^2}{r}$. cos. B=$\frac{v^2}{r}$. sine of the inclination of plane of osculating circle to the plane which touches the surface.

(aa) Let the perpendicular distances of the given points from the plane which separates the two media $= a, a'$, if through these two points a plane be conceived to pass perpendicular to the plane surface which separates the media, and if the line described be supposed to be projected on this plane, then, since the extreme points of this line are given, a a' the perpendicular distances of these points from the separating plane will also be given; and also c the intercept between these perpendiculars reckoned on this plane, let x, x', denote the angles which the projection of the line on the perpendicular plane makes with the perpendicular to the separating plane at the point, where the projection of the line described meets the separating plane; then we have evidently $c = a$ tang. $x + a'$ tang. x', if z denotes the perpendicular distance of the point where the ray of light meets the separating plane from its projection on the perpendicular plane, and y y' the distances of the given points from this plane, we have evidently $y = \sqrt{z^2 + \frac{a^2}{\cos.^2 x}}$, $y' = \sqrt{\left(z^2 + \frac{a'^2}{\cos.^2 x'}\right)}$; but as the density of the two media through which the light passes,

though different, from one to the other, is uniform for each of them respectively; n n' the velocities in those media will be uniform, ∵ $\int v\,d\,s = n\,y$ is the part of the integral of $v\,d\,s$ which appertains to the first medium, and $n'\,y'$ the part of this integral which appertains to the second, consequently by Note (*bb*) $ny + n'\,y' = \int v\,d\,s$ is a minimum, *i. e.* $n.\sqrt{z^2 + \frac{a^2}{\cos.^2 x}} + n'\sqrt{z^2 + \frac{a'^2}{\cos.^2 x'}}$ is a minimum with respect to z, x, x', of these x x' are connected by the equation $c = a.\tan.x + a'.\tan.x'$; ∵ in the first place the differential of the preceding function with respect to $z = o$, *i. e.* $n.\frac{dy}{dz} + n'.\frac{dy'}{dz} = o$, but $\frac{dy}{dz} = \frac{z}{y}$, $\frac{dy'}{dz} = \frac{z}{y'}$, ∵ $\frac{nz}{y} + \frac{n'z}{y'} = o$, but as this equation cannot be satisfied unless $z = o$, it follows, that the track of the luminous ray coincides with the plane perpendicular to the plane separating the surfaces, and passing through the two given points; therefore

$$ny + n'y' = \frac{a\,n}{\cos.x} + \frac{a'\,n'}{\cos.x'}$$ which as it is a minimum,

$$\frac{a\,n.\sin.x.}{\cos.^2 x}dx + \frac{a'n'.\sin.x'}{\cos.^2 x'}dx' = o,$$ but differentiating the equation $c = a.\ \tan.x + a'.\tan.x'$ we obtain

$$\frac{a.\,dx}{\cos.^2 x} + \frac{a'.\,dx'}{\cos.^2 x'} = o,$$ hence eliminating $\frac{dx'}{dx}$ between these two equations we find $n.\sin.x = n'\sin\ x'$; but x is the angle of incidence, and x' the angle of refraction, *whose sines are therefore in a given ratio.* If the ray of light instead of penetrating the second medium is reflected back, then the velocity remains the same during the entire route, and $\int v\,d\,s$ becomes $v\int d\,s$, which is by hypothesis a minimum: therefore the track of the ray is the shortest possible, consequently it makes = angles with the reflecting surface, ∵ the angles of incidence and reflexion are =.

(*bb*) $v^2 = C + 2.\int(Pdx + Qdy + Rdz)$ *see* page 432 ∵ $v\delta v = P\delta x + Q\delta y + R\delta z$, ∵ substituting in the equation of page 431; we have $\delta x\,d.\frac{dx}{dt} + \delta y\,d.\frac{dy}{dt} + \delta z\,d.\frac{dz}{dt}. = v.\ dt.\ \delta v =$

$ds.\ \delta v$, now as $ds = \sqrt{dx^2+dy^2+dz^2}$, $\frac{ds}{dt}.\ \delta\, ds =$

$\frac{dx}{dt}\ \delta\, dx + \frac{dy}{dt}.\ \delta\, dy + \frac{dz}{dt}.\delta\, dz$, and as it is indifferent which of the characteristics d or δ precedes the other; we have

$\frac{ds}{dt}.\ \delta\, ds = v.\ \delta\, ds = d.\left(\frac{dx\ \delta x + dy\ \delta y + dz.\ \delta z}{dt}\right) - \delta x\ d.\frac{dx}{dt} -$

$$\delta y.\ d\frac{dy}{dt} - \delta z\, d\frac{dz}{dt}\ \because v.\ \delta\, ds + \delta v.\ ds = \delta\,(vds) =$$

$d.\left(\frac{dx.\ \delta z + dy\ \delta y + dz.\ \delta z}{dt}\right)$; integrating with respect to d. we have $\delta \int (v\ d\ s) = C' +$

$\frac{dx\ \delta x + dy\ \delta y + dz\ \delta z}{dt}$; when the extreme points of the line described by the point are fixed, $\delta x\ \delta y\ \delta z$ are $=$ to cypher at these points; $\therefore\ \delta \int(v.\, ds) = o$ for C' evidently vanishes; $\therefore$ $\int(v\ d\ s)$ is either a maximum or minimum : but it is evident from the nature of the function $\int(v\ d\ s)$ that it is not a maximum; hence of all curves which a point sollicited by the forces P Q R, describes in its passage from one given point to another, it describes that in which $\delta.\ (v\ d\ s) = o$, consequently that in which $v.\ ds$ is a minimum; if there are no accelerating forces v is constant, and $\int(v.\ ds)$ becomes $v.\int d\ s$, $\therefore$ in this case the curve described by the moving point is the shortest, and in consequence of the uniformity of the motion the time will also be a *minimum* : since $\delta\int v.\ ds = o$, is true in all cases in which $Pdx + Qdy + Rdz$ is an exact differential, it is true for all curves described by the actions of forces directed to fixed centres, the forces being $\div\!\!\div$l to functions of distances from them ; and if the form of these functions was given, we could determine the species of the curve described, by substituting for v its value in terms of the force, and then investigating by the calculus of variations the relation between the ordi-

nates of the curve, which satisfies $\delta\,(vds)=o$. If the force varied as $\frac{1}{s^2}$ it would be easy to shew that the curve was a conic section origin in the focus, if the force varied as s the distance from centre, the curve described would be also a conic section, origin being in the centre.

CHAPTER III.

(*a*) In fact let p denote the action which m the first exerts on m' the second, if previous to the impact, m' is actuated by p and $-p$; the first m is employed in destroying $-p$, and to effect this it must employ a force $=$ and directly contrary to $-p$, and therefore it will lose a force $=$ to p;

(*b*) g the gravity must, however, be distinguished from w the weight, for g denotes the intensity of the power as it exists in nature without any reference to the quantity of matter put in motion; w denotes the force of gravity applied to the particular body under consideration, which depends not only on the intensity of the gravity, but also on the mass of the body on which it is exerted, so that w is the resultant of all the forces of gravity acting on each molecule. w is $\div$l to m, the quantity of matter, at a given place, but to determine the value of w in different latitudes, we must take into account the intensity of gravity, which varies from one place to another, $\therefore w = mg$ and as $m = v\,d$, $w = v.\,d.\,g.$ v being the volume and d the density.

(*c*) The reason why distilled water was selected as the term of comparison was, that it was one of the most homogeneous substances, and the maximum of its condensation

was easily ascertained, as it always obtained about 4° above the freezing point the centigrade thermometer.

(*d*) What is here stated does not in the least tend to establish the exploded position of Des Cartes, that all space was equally full of matter, for according to him, all matter was homogeneous, and the subtle ether which was diffused through the planetary regions was of the same nature with other matter.

(*e*) Since perpendiculars from any point in the direction of the resultant of two forces, on the directions of the forces, are inversely as the forces, it follows that as in this case the resultant passes through the fulcrum, perpendiculars from fulcrum on the directions of the composing forces, are inversely as the forces.

(*f*) In general it may be remarked that the *whole* force necessary to perform any work is not diminished by the application of the mechanic powers, their use is either to diminish the force applied at once by lengthening the time, or to shorten the time, by increasing the force applied at once.

(*g*) This will immediately appear from Notes to Chapter II, for V the resultant resolved parallel to the axis of $x = \text{V}.\left(\frac{x-\text{A}}{u}\right)$, $=$ (as $x = \rho.\cos.\psi$, ρ being the projection u on the plane of $x\,y$) $\text{V}.\left(\frac{\rho.\cos.\psi-\text{A}}{u}\right)$ and this force resolved in the direction perpendicular to ρ *i. e.* in the direction of

$$\rho\,\delta\,\psi = \frac{\text{V}.}{u}(\rho.\cos.\psi-\text{A}).\frac{y}{\rho} = \frac{\text{V}.}{u}(\rho.\cos.\psi-\text{A}).\sin.\psi,$$

in like manner V when resolved parallel to the axis of y, and then perpendicular to ρ or in the direction of $\rho\,d\,\psi$ $= \frac{\text{V}.}{u}(\rho.\sin.\psi-\text{B}).\cos.\psi$, $\because$ the efficient part of V resolved in the direction of the element $\rho\,\delta\,\psi =$ $\frac{\text{V}}{u}.((\rho.\sin.\psi-\text{B}).\cos.\psi-(\rho.\cos.\psi-\text{A})\sin.\psi)$, which as u^2

$=(\rho.\cos.\psi-A)^2+(\rho.\sin.\psi-B)^2+(z-C)^2$, and $\therefore u\left(\frac{\delta u}{\delta\psi}\right)$ $=-\rho.\sin.\psi\,(\rho.\cos.\psi-A)+\rho.\cos.\psi\,(\rho.\sin.\psi-B)$ is $=$ to $\frac{V}{\rho}.\left(\frac{\delta u}{\delta\psi}\right)=\frac{p\,V'}{\rho}$; V' being the projection of the given force on the plane xy, and p a perpendicular from the axis of z on the direction of V', and $\therefore\frac{p.V'}{\rho}$, the projected force resolved in a direction perpendicular to ρ, therefore we have $V.\left(\frac{\delta u}{\delta\psi}\right)=p\,V'=$ the moment of the projection of V with respect to the origin, but $V.\left(\frac{\delta u}{\delta\psi}\right)=\Sigma.S.\left(\frac{\delta s}{\delta\psi}\right)=$ the sum of moments of the composing forces, *see* page 410.

(*g*) It appears from the expression $p.V'$ that the moment of a force may be geometrically represented by means of a triangle, whose vertex is at the point, and whose base represents the intensity of the force; and if X, Y indicate the force V, resolved parallel to the axes of x,y respectively, $X=V_{,}.\left(\frac{x-A}{u}\right)$, $Y=V_{,}.\left(\frac{y-B}{u}\right)$, and these forces resolved respectively perpendicular to ρ, are $V_{,}.\left(\frac{x-A}{u}\right).\frac{y}{\rho}$, $V_{,}.\left(\frac{y-B}{u}\right).\frac{x}{\rho}$; their difference $=\frac{Yx-Xy}{\rho}=\frac{p.V'}{\rho}$.

(*h*) Hence if either p or V', vanish the moment is $=$ to o, and as the projection of the area of a plane curve on another plane, is equal to this area multiplied by the cosine of the angle contained between the two planes, it follows that the moment of the forces relative to any axis inclined to the greatest moment is equal to the greatest moment multiplied into the cosine of this inclination.

(*i*) If $s=$ the inclination of two planes of the moments H and V_0, or which is the same thing, the inclination of two perpendiculars to these planes; and if a,a',a'',b,b',b'', represent the angles which these perpendiculars make respectively with

three rectangular axes, $\cos. s = \cos. a. \cos. b + \cos. a'. \cos. b' + \cos. a'' \cos. b''$; $\therefore$ when $s = 90$, this function $= o$; we have also $\cos.^2 a + \cos.^2 a' + {}^2 a'' = 1$; $\cos.^2 b + \cos.^2 b' + \cos.^2 b'' = 1$; $\therefore$ if $V_{,}$ $V_{,,}$ $V_{,,,}$ represent the projections of the given moment H on three rectangular planes, xy, xz, yz, we have $V_{,} = H. \cos. a$, $V_{,,} = H. \cos. a'$, $V_{,,,} = H. \cos. a''$; in like manner we have $V_o = H. \cos. s = H. \cos. a. \cos. b + H \cos. a'. \cos. b' + H. \cos. a''. \cos. b'' = V_{,}. \cos. b + V_{,,}. \cos. b' + V_{,,,}. \cos. b''$; $\therefore$ if we know the projection of the greatest moment on any three rectangular planes, we have its projection on any plane whose inclination to those is given; in like manner, if $V_{o,}$ $V_{o,,}$ represent the projections of H on two planes rectangular to each other and to the plane of projection of V_o, $b_{,}$ $b_{,}'$ $b_{,}''$ $b_{,,}$ $b_{,,}'$ $b_{,,}''$ being the angles which perpendiculars to these planes make respectively with x y z, we have $V_{o,} = V_{,}. \cos. b_{,} + V_{,,}. \cos. b_{,}' + V_{,,,}. \cos. b_{,}'''$, $V_{o,,} = V_{,}. \cos. b_{,,} + V_{,}. \cos. b_{,,}' + V_{,}. \cos. b_{,,}''$, $\therefore$ it follows, that $V_{,}^2 + V_{,,}^2 + V_{,,,}^2 = V_o^2 + V_{o,}^2 + V_{o,,}^2$; hence it appears that $V_{,}^2 + V_{,,}^2 + V_{,,,}^2$ is independent of the direction of the three perpendicular planes of projection, and $V_o = \sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2 - V_{o,}^2 - V_{o,,}^2}$; $\therefore$ V_o is a maximum and $=$ H *i.e.* $\sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2}$ when $V_{o,} = o$, $V_{o,,} = o$, $\therefore$ this constant quantity is the value of the maximum moment, and $V_{,} = V_o. \cos. a$, $V_{,,} = V_o \cos. a'$, $V_{,,,} = V_o. \cos. a''$, $\therefore$ $\cos. a =$

$$\frac{V_{,}}{\sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2}}, \quad \cos. a' = \sqrt{\frac{V_{,,}}{V_{,,,}^2 + V_{,,}^2 + V_{,,,}^2}}, \quad \cos. a'' =$$

$$\frac{V_{,,,}}{\sqrt{V_{,}^2 + V_{,,}^2 + V_{,,,}^2}};$$ $\therefore$ if we know the moments with respect to three rectangular planes arbitrarily selected, we have the value of the principal moment, and also its position; and if on perpendiculars to each of these three planes, lines be assumed respectively $\div$ l to the projections of the moments on these planes, the diagonal of the parallelopiped, of which, these three lines are the sides, represents the maximum moment in quantity and direction.

(*k*) Therefore it appears from Note (*i*) there will be an equilibrium, if the principal moment and the resultant of all the forces = cypher respectively; if there is a fixed point in the system, the resultant of all the forces is destroyed by its reaction; if there is no fixed point the resultant V must vanish, but this cannot be the case, unless each of the forces X Y Z respectively vanish; as $Xy - Yx = V_{,}$ so it might be shewn that $Zx - Xz = V_{,,}$; $Yz - Zy = V_{,,,}$; but as these three equations obtain at the same time, we have by multiplying the first by Z, (*see* Celestial Mechanics, page 89,) the second by Y, and the third by X, and then adding them together $V_{,}Z + V_{,,}Y + V_{,,,}X = o$, this is the =n of condition, which must be satisfied when the forces have an unique resultant; if X Y Z are = respectively to cypher, then the forces are reducible to two respectively =, but not directly opposed to each other.

(*l*) It is evident from what has been established in Notes (*g*) (*h*) of this Chapter, that generally the sum of the three composing forces, parallel to the three rectangular coordinates, are $\Sigma.\ m\ S.\ \left(\frac{\delta s}{\delta x}\right)$, $\Sigma.\ m.\ S.\ \left(\frac{\delta s}{dy}\right)$, $\Sigma\ m\ S\left(\frac{\delta s}{\delta z}\right)$; and the sum of the moments projected on the three planes may be expressed thus:

$$\Sigma\, m\ S\left(y.\left(\frac{\delta s}{\delta x}\right) - x.\left(\frac{\delta s}{\delta y}\right)\right);\ \Sigma.m.S.\left\{z.\left(\frac{\delta s}{dx}\right) - x\left(\frac{\delta s}{\delta z}\right)\right\};$$

$\Sigma\ m\ S\left\{y.\left(\frac{\delta s}{\delta z}\right) - z.\left(\frac{\delta s}{\delta y}\right)\right\}$, in the case of equilibrium, and that the point is free, these quantities are respectively = to cypher; if the forces acting on the system be those of gravity, $S = S', = S''$, &c.$\frac{\delta s}{\delta x} = \frac{\delta s}{\delta x'} = \frac{\delta s}{\delta x''}$, &c. the first three equations become $S.\left(\frac{\delta s}{\delta x}\right).\ \Sigma m$, $S.\left(\frac{\delta s}{\delta y}\right)\Sigma\ m$, $S.\left(\frac{\delta s}{\delta z}\right)\Sigma\ m$, and the last three become $S.\left(\frac{\delta s}{dx}\right).\ \Sigma\ m\ y - S.\left(\frac{\delta s}{\delta y}\right).\ \Sigma\ m\ x$;

$S.\left(\frac{\delta s}{\delta x}\right).\ \Sigma\, m\, z - S.\left(\frac{\delta s}{\delta z}\right).\ \Sigma\, m\, x$; $S.\left(\frac{\delta s}{\delta z}\right).\Sigma\, m\, y - S.\frac{\delta s}{\delta y}.$ $\Sigma\, m.\, z$; and the three first compound a unique force $= S.\Sigma\, m$ *i.e.* the weight of the system, which is destroyed by the reaction of the origin when it is fixed. If the origin of the coordinates be a given point different from the centre of gravity, and if C B A be the coordinates of the centre with respect to this point; then $\Sigma m.(x-A)=o$; $\Sigma m.(y-B)=o$; $\Sigma m.(z-C)=o$ when the origin is *fixed*; $\because$ we have $A.\Sigma m = \Sigma mx$; $B.\ \Sigma m = \Sigma my$; $C.\ \Sigma m = \Sigma mz$; hence knowing the positions of the several bodies of the system with respect to the axes of x, y, z, we can determine the coordinates of the centre of gravity with respect to the same axes. As $(\Sigma(mx))^2 = \Sigma(m^2x^2) + 2\,\Sigma(mm', xx')$; and $\Sigma mm'(x-x')^2 = mm'x^2 + m\,m'x'^2 + m\,m''x^2 + m\,m''^2 + m'm''x''^2 + \&c. - 2mm''\,xx'' - 2m'\,m''\,x'x'' - \&c. = \Sigma(m\,m'x^2) - 2\Sigma(mm'\,xx')$; and as $\Sigma(mx^2).\Sigma m = \Sigma(m^2x^2) + \Sigma(m\,m'\,x^2)$, $\because \Sigma(mx)^2 = \Sigma(mx^2)\Sigma m - \Sigma m\,m'x^2 - \Sigma mm'(x-x')^2 + \Sigma(mm'x^2)$, $\because A^2 =$

$$\frac{(\Sigma mx)^2}{\Sigma m^2} = \frac{(\Sigma mx^2)}{\Sigma m} - \frac{\Sigma m\,m'(x-x)^2}{(\Sigma m)^2},$$

we might obtain corresponding values for B^2 and C^2, hence it is evident that $A^2+B^2+C^2 =$

$$\frac{\Sigma m.(x^2+y^2+z^2)}{\Sigma m} - \frac{\Sigma mm'((x'-x)^2+(y'-y)^2+(z'-z)^2}{(\Sigma m^2)}$$

consequently if we have the distances of the several bodies of a system from a given point and also their mutual distances from each other, we have the distance of the centre of gravity of those bodies from the same point, and if the same be given for three fixed points, the position of the centre of gravity in space will be obtained. If the expression $\Sigma((x-A)^2+(y-B)^2+(z+C)^2)$ be differentiated with respect to $x\,y\,z$ respectively, and the differential coefficient be then put $= o$, we shall have $\Sigma(x-A)=o$, $\Sigma(y-B) = o$, &c. this implies that the sum of the squares of the distances of the molecules from the point ABC is a minimum,

and if these molecules are all equal to each other, and represented by m, we have $\Sigma m(x-A)=o$, $\Sigma m(y-B)=o$, &c.

$$\therefore A = \frac{\Sigma mx}{\Sigma m},\ B = \frac{\Sigma my}{\Sigma m},\ C = \frac{\Sigma mz}{\Sigma m};$$

consequently the centre of gravity of a system possesses this property, namely, that the sum of the squares of the distances of the points of the system from it, is less than for any other point whatever. If several forces concurring in a point constitute an equilibrium, and if at the extremities of lines ÷l to and in the directions of these forces, be placed the centres of gravity of = bodies, the common centre of gravity of these bodies will be the point where the forces concur; for as the forces are represented by lines taken in their direction and concurring in one point, if this point be made the origin of the coordinates, the sum of the forces parallel to the axes of $x\ y\ z$ are $\Sigma(x), \Sigma(y), \Sigma(z)$, and by hypothesis they are = to cypher, $\therefore \Sigma(x), =o$, $\Sigma(y)$ $\Sigma(z)=o$, *i. e.* since the bodies are equal $\Sigma(mx), \Sigma(my), \Sigma(mz)$, are = to cypher, consequently the origin is in the centre of gravity of a system of bodies of which each is equal to m, $\therefore$ if to all the points of any body, forces be applied directed towards the centre of gravity, and ÷l to the distances between these points and the centre of gravity, these forces constitute an equilibrium; it likewise appears that when several forces constitute an equilibrium, the sum of the squares of the distances of the point of concourse of these forces from the extremities of lines ÷l to these forces, is a minimum.

(*m*) This principle was established first by a copious induction of particular cases; it may be thus analytically announced, if S, S′, S″, &c. represent the forces actuating the several points of the system and δs, $\delta s'$, $\delta s''$, &c. the spaces moved over in the respective directions of these forces, we have $m\ S\ \delta s + m'\ S'\ \delta s' + m''\ S''\ \delta s'' + \&c. = o$ in the case of the equilibrium of the system, the variations

being subjected to the condition of the connexion of the parts of the system.—(*See* page 411, and also Celestial Mechanics, page 82.) It is also evident, that if the preceding equation obtains the system is in equilibrio; for suppose that while the preceding equation obtains the points m, m', m'', &c. are actuated by the velocities v, v', v'', in consequence of the action of the forces m S, m'S$'$, m''S$''$, &c. which are applied to them, the system will evidently be in equilibrio, in consequence of the action of these forces, and of $m\,v$, $m'\,v'$, $m''\,v''$, &c. applied in a contrary direction, $\because \delta v$, $\delta v', \delta v''$, &c. denoting the variations of the directions of the new forces, we shall have from the preceding principle, $m\text{S}.\delta s + m'\text{S}\ \delta s' + m''\ \text{S}''\ \delta s'' + \text{\&c.} - mv\delta v - m'v'\delta v' - m''v''\delta v'', \text{\&c.} = o$, but the positive part of this equation vanishes by hypothesis, $\because mv\delta v + m'v'\delta v' + m''v''\delta v''$, &c. $= o$, if we assume $\delta v = vdt$, $\delta v' = v'dt$, $\delta v'' = v''\delta t$, &c. as we are permitted to do, we shall have $mv^2 + m'v'^2 + m''\,v''^2 + \text{\&c.} = o$; $\because v = o$, $v' = o, v'' = o$, &c.; *i. e.* the system is in equilibrio when $m\,\text{S}\delta s + m'\ \text{S}'\,\delta s' + m''\ \text{S}''\delta s''$, &c. $= o$;

The condition of the connexion of the parts of the system may be reduced to equations between the coordinates of the several bodies, if $u' = o$, $u'' = $o, &c. be these different equations, we should add, as in page 412, $\lambda\ \delta u, \lambda'\ \delta u'$, &c. to the function $\Sigma m.\ \text{S}\,\delta s$; λ, λ', &c. being indeterminate functions which should be determined in the manner suggested in page 412, the equation given above then becomes $\text{o} = \Sigma m\ \text{S}\delta s + \Sigma\lambda\ \delta u$; in this case we may treat the variations of the coordinates as arbitrary, and $\because$ put their respective coefficients $= o$; which will give as many distinct equations, and thus enable us to determine $\lambda\ \lambda'$ &c. and therefore R R$'$ &c.

The six equations of equilibrium which were given in page 422 may be deduced from the equation $o = \Sigma m\text{S}ds$, for if the bodies of the system are firmly united to each other, their mutual distances $d\ d'\ d''$ &c. are invariable, and

$$\because \sqrt{(x'-x)+(y'-y)^2+(z'-z)^2}$$
$$\sqrt{(x''-x)^2+(y''-y)^2+(z''-z')^2}$$
$$\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}$$

are constant, and therefore their variations $= o$; $\because$ whatever suppositions satisfy these conditions, will also obtain for the equation $\Sigma m.\ \mathrm{S}\,\delta s$, but these variations $= o$, from either of two suppositions, namely, either from making $\delta x = \delta x' = \delta x''$, &c. $\delta y = \delta y' = \delta y''$, &c. $\delta z = \delta z' = \delta z''$, &c. or from making $\delta x = y\delta\varpi$, $\delta y = -x\delta\varpi$ $\delta x' = y'\ \delta\varpi$, $\delta y' = -x'\delta\varpi$, &c. $\delta\varpi$ being any variation whatever, as is evident from making these suppositions in the variations of d, d', d'', &c. in the first case it is evident that these substitutions make

$$m\ \mathrm{S}.\frac{\delta s}{\delta x}+m'\mathrm{S}'\frac{\delta s'}{\delta x}+m''\mathrm{S}''\frac{\delta s''}{\delta x}+\&c. = o; \quad i.\ e.$$

$\Sigma m\,S\dfrac{\delta s}{\delta x} = o$, $\Sigma m\mathrm{S}\dfrac{\delta s}{\delta y} = o$; $\Sigma m\ \mathrm{S}.\ \dfrac{\delta s}{\delta z} = o$; which imply that in the case of the equilibrium of a system of bodies, the sum of the forces of the system resolved parallel to the axes of $x\ y\ z$ are $=$ to cypher. By substituting the other values of $\delta x\ \delta y$ &c. which satisfy the condition of the invariability of the distances of the bodies of the system, in the equation $o = \Sigma m.\ \mathrm{S}\delta s$, we obtain $o = \Sigma m\mathrm{S}.\left(y\dfrac{\delta s}{\delta x} - x\dfrac{\delta s}{\delta y}\right)$, and by changing the coordinates x, x', x'', &c. or y, y', y'', &c. into z, z', z'', &c. we shall obtain $o = \Sigma m\mathrm{S}\left\{y\left(\dfrac{\delta s}{\delta z}\right) - z\left(\dfrac{\delta s}{\delta y}\right)\right\}$; $o = \Sigma\,m\,\mathrm{S}.\ \left\{z\left(\dfrac{\delta s}{\delta x}\right) - x\left(\dfrac{\delta s}{\delta z}\right)\right\}$; which are the three other equations of the equilibrium of a system, indicating that the sum of the moments of the forces, parallel to any two axes, which would cause the system to revolve about the remaining axis, are respectively equal to cypher; if the origin of the coordinates was fixed and at-

tached invariably to the system, the forces parallel to the three axes will be destroyed by the reaction of the fixed point, so that the three last equations are those which remain to be satisfied ; in this case the resultant of all the forces which act on the body, passes through the fixed point, and therefore is destroyed by its reaction. If there are two points fixed in the system there is only one equation of equilibrium, namely, that which expresses that the sum of the moments of the forces, which would make the system revolve about the line connecting the fixed points as an axis, is equal to cypher; in general the number of equations of equilibrium is equal to the number of possible motions which can be impressed on the system, or to the least number of indeterminate quantities.

(*o*) Perhaps it would be more accurate to say, that there were three kinds of equilibrium, namely, stable, unstable, and neutral, in the last the body is in a state of indifference, and has no tendency either to recover its primary position, or to deviate more from it; it is evident that it only obtains when the equilibrium exists under a continued change of position, a homogeneous sphere, or a cylinder whose axis is horizontal, floating in a fluid, are instances of this species of equilibrium, for they have no tendency to maintain one position in preference to another.—*See* Note (*h*) Chapter IV, and Notes (*f*) (*k*) Chapter V.

CHAPTER IV.

(*a*) By Notes, page 411, it appears that when the fluid is at rest, and ∵ each molecule at rest, the resultant of all the forces, by which it is actuated, must be at right angles to the surface on which it exists; for it follows from the perfect mobility of the particles of fluids, that when a fluid mass is in equilibrio, each constituent molecule of the fluid must also be in equilibrio. When a molecule exists on the surface of the fluid, the resultant of all the forces by which it is actuated, must from what is already established, be perpendicular to that surface; a molecule in the interior of the fluid mass is subjected to two distinct actions, one arising from the forces which solicit it, and the other from the pressure produced by the surrounding particles; and the entire pressure at any point arises from the combined action of the two. If a fluid mass, of which the molecules are solicited by any accelerating forces whatever, be in equilibrio, when contained in a vessel, which is closed on *every* side; and if the equilibrium would cease to exist if an aperture was made in the side of the vessel, it is necessary in this case, that the pressure exerted on the sides, should be perpendicular to them, as otherwise the resistance of the surface would not destroy the pressure; the intensity of this pressure in general varies from one point to another, and depends on the accelerating forces and on the position of the point; with respect to those fluids, which are termed elastic, they may indeed press on the sides of the vessel in which they are enclosed, though no motive forces act on the particles, or without any pressure urging the surface of the fluid; for as they perpetually endeavour to dilate themselves in consequence of their elasticity, this gives

rise to a pressure on the sides of the vessel, which is always constant for the same fluid, and depends in general on the matter of the fluid, its density and temperature.

(*b*) By considering each molecule as an indefinitely small rectangular parallelopiped, we are permitted to suppose that the pressure accelerating forces and density of each point of any one of its surfaces are the same; we also can thus express the fact of the equality of pressure which is the fundamental principle from which the whole theory of their equilibrium may be deduced; let p denote the mean of all the pressures on any side, p' the corresponding pressure on the opposite side, ρ the density, P, Q, R, the accelerating forces which solicit the molecule, resolved parallel to the three coordinates of the angle of the parallelopiped next to the origin; then $dx\ dy\ dz$ represent the dimensions of the parallelopiped, and p being a function of $x\ y\ z$, we have $\delta p = \frac{\delta p}{\delta x}.\ \delta x + \frac{\delta p}{\delta y}.\ \delta y + \frac{\delta p}{\delta z}.\ \delta z$; now the parallelopiped, in consequence of the *pressure* to which it is subjected, will be urged in the direction of x by the force $(p'-p).\ dy.\ dz$, but as $p'-p$ is the differential of p, taken on the hypothesis that x only is variable, we have $p' - p = \frac{dp}{dx}dx = \frac{\delta p}{\delta x}.\ dx$; $\therefore (p-'p).\ dy.\ dz = -\frac{dp}{dx}\ dx.\ dy.\ dz$, (we have taken $\frac{dp}{dx}$, &c. negatively, because they diminish the coordinates); but ρ being the density of the molecule, its quantity of matter $= \rho.\ dx.\ dy.\ dz.$ and its *motive force* arising from P, $= \rho$ P. $dx\ dy\ dz$; the force with which the molecule is solicited in consequence of the action of this force and of the pressure, both of which act on the molecules $= \left\{\rho\ \mathrm{P} - \left(\frac{dp}{dx}\right)\right\} dx.\ dy.\ dz$; similar equations may be obtained for the forces parallel to y and z.

(c) By hypothesis the molecule is in equilibrio, therefore in consequence of what is established in Notes, page 430, we have

$$o = \left\{\rho\,P - \left(\frac{dp}{dx}\right)\right\}.\,\delta x + \left\{\rho\,Q - \left(\frac{dp}{dy}\right)\right\}.\,\delta y +$$

$\left\{\rho\,R - \left(\frac{dp}{dz}\right)\right\}.\,\delta z$, *i. e.* $\delta p = \rho(P.\delta x + Q.\,\delta y + R.\,\delta z)$; as δp is an exact variation, the second member must be so likewise, therefore

$$\frac{d.\,\rho\,P}{dy} = \frac{d.\,\rho.Q}{dx};\quad \frac{d.\,\rho.\,P}{dz} = \frac{d.\,\rho.\,R}{dx};\quad \frac{d.\,\rho.\,R}{dy} = \frac{d.\,\rho.\,Q}{dz}$$

and multiplying the first of these equations by R, the second by — Q, and the third by P, we obtain by expanding,

$$\frac{\rho.\,R.\,dP}{dy} + \frac{R.\,P.\,d\rho}{dy} = \frac{R\rho.\,dQ}{dx} + \frac{RQ.\,d\rho}{dx};\quad -\frac{\rho.\,Q.\,dP}{dz} -$$

$$\frac{Q.\,P.\,d\rho}{dz} = -\frac{\rho Q.\,dR}{dx} - \frac{RQ.\,d\rho}{dx};\quad \frac{\rho P.\,d.Q}{dz} + \frac{P.Q.\delta\rho}{dz} = \frac{\rho P.\,dR}{dy}$$

$+\dfrac{RP.\,d\rho}{dy}$ by reducing all the terms of which $\delta\rho$ is a factor to one side, and adding them together, we obtain

$$\rho.\left\{\frac{R.\,dP}{dy} - \frac{R.\,dQ}{dx} - \frac{Q.\,dP}{dz} + \frac{Q.\,dR}{dx} + \frac{P.\,dQ}{dz} - \frac{P.\,dR}{dy}\right\}$$

$$= \left(-\frac{RP}{dy} + \frac{RQ}{dx} + \frac{QP}{dz} - \frac{R\,Q}{dx} - \frac{P\,Q}{dz} + \frac{RP}{dy}\right)\delta\rho = o;$$

therefore by concinnating

$$P.\,\frac{dQ}{dz} - Q.\,\frac{d\,P}{dz} + R\frac{d\,P}{dy} - P.\frac{dR}{dy} + Q.\frac{dR}{dx} - R.\frac{dQ}{dx} = o;$$

when this equation is satisfied, the equilibrium obtains though ρ remains undetermined. But if the relation indicated by this equation does not obtain between the forces P, Q, R, the fluid will be in a perpetual state of agitation whatever figure it may be made to assume; and if P Q R be functions of the coordinates, $\rho(P\delta x + Q\delta y +$

$R\delta z$) can be integrated by the method of quadratures, by means of which we can find the value of p for any given point, and ∴ the force with which any side of the vessel is pressed; but though the equilibrium be impossible when the equation of condition is not satisfied, it does not follow that when it is satisfied that the equilibrium will obtain; for in most cases this fluid must assume a determined figure depending on the nature of P Q R. Likewise though in the state of equilibrium all the molecules in the same strata have necessarily the same density, and experience the same pressure, the converse is not true, for in homogeneous incompressible fluids ρ is constant, in those sections of the fluid in which neither $\delta\phi$ nor $\delta p = o$;

δp never $= o$, if the fluid be elastic, because ρ being a function of p, if the density has a finite value, p can never vanish.

(d) If the fluid be free at its surface, $p = o$, ∴ if $\delta x\ \delta y\ \delta z$ belong to the surface we have $o = P\delta x + Q\delta y + R\delta z = \lambda du$, u being the equation of the surface, and λ a function of $x\,y\,z$; therefore the resultant of P Q R must be perpendicular to those parts of the surface in which the fluid is free, for in this case

$$\frac{P}{\sqrt{P^2+Q^2+R^2}} \quad \frac{Q}{\sqrt{P^2+Q^2+R^2}}, \quad \frac{R}{\sqrt{P^2+Q^2+R^2}}$$

are the cosines of the angles which the resultant of P Q R makes with the axes of $x\ y\ z$, but as $P\delta x + Q\delta y + R\delta z = \lambda\delta u$ they express also the cosines of the angles which the same axes make with the normal, ∴ the normal coincides with the resultant; this coincidence of the normal with the resultant is a condition which must also be satisfied to insure the equilibrium; by means of it we can determine in each particular case, the figure corresponding to the equilibrium of the fluid; if for instance there is only one attractive force directed towards a fixed point, the form of the surface will be spherical,

the fixed point being the centre of the sphere; (if this point be at an infinite distance the sphere degenerates into a plane,) for in this case $\frac{dp}{\rho}=\frac{F}{r}\left\{(x-a)\,dx+(y-b)dy+(z-c).dz\right\}=F.\,dr$; a b c are the coordinates of the centre, to which F the force is directed, and if the origin of the coordinates be in the centre, we have $\frac{dp}{\rho}=\frac{F}{r}(xdx+ydy+zdz)$, if the density ρ is constant, or a function of p, the equation of each stratum of level becomes $x^2+y^2+z^2=C$, which belongs to a sphere of which the centre coincides with the point of common reunion of the directions of all the forces.

When $P\delta x+Qdy+R\delta z$ arise from attractive forces, as is stated in the text, it must be an exact variation $=\delta\phi$, $\therefore$ we have $\delta p=\rho.\,\delta\phi$, consequently as $\frac{\delta p}{\rho}=\delta\phi$, ρ must be a function of p, therefore p will be a function of ρ, and they will be same for all those molecules in which the value of ϕ is given, *i. e.* for molecules of the *same strata of level*; and for a fluid in which ϕ varies, an equilibrium cannot take place unless each respective stratum be homogeneous, for in this case ρ and $\therefore$ p is constant; for surfaces in which ρ is constant $\delta p=o$, therefore for such surfaces $P.\,\delta x+Q.\,\delta y+R.\,\delta z=o$, and the resultant coincides with the normal. The integral of $o=\rho\delta\phi$ is a constant arbitrary quantity, which indicates that the given equation $\phi=C$ appertains to an indefinite number of surfaces differing from each other by the value which is assigned to this quantity; in the equation $\phi=C$, $d\phi$ evidently $=o$, $\therefore$ ϕ is either a *maximum* or *minimum*, and generally when $P\delta x+Q\delta y+R\delta z$ is an exact variation, ρ is a function of ϕ, $\therefore$ the equation $\delta p-\rho.\,\delta\phi=o$, indicates that in the state of equilibrium, there is a function of p and x y z, which is either a maximum or minimum. If this quantity be increased by insensible gradations, we shall have an indefinite number of

surfaces, distributing the entire mass into an indefinite series of strata constituting between any two strata what have been termed *strata of level*; the law of the variation of density ρ in passing from one stratum to the consecutive one, is altogether arbitrary, as it depends on what function of ϕ, ρ is, but this is undetermined. It appears, therefore, from what precedes, that there are two cases in which $\delta p = o$, when it is at the free surface, in which case p vanishes of itself, and also when p is constant, *i. e.* for all surfaces of the same level; $\because$ when the fluid is homogeneous, the strata to which the resultant of the forces is perpendicular, are necessarily of the same density; when the fluid is contained in a vessel closed on every side, it is only necessary that all strata of the same level should have the same density; in elastic fluids it never could happen that p should vanish, or that $P\delta x + Q\delta y + R\delta z = o$, therefore unless the fluid extends indefinitely into space, so that ρ may be altogether insensible, it cannot be in equilibrio except in a vessel closed on every side.

(*d*) In the case of our atmosphere ρ is observed to be $\div$l to p *i. e.* $p = k\rho$, k depends on the temperature and matter of the fluid, by substituting for ρ in the equation $\delta p = \rho\delta\phi$ we obtain $\delta p = \frac{p}{k}.\delta\phi$ $\because$ by integrating $\log. p + C = \frac{\phi}{k}$, because when the temperature and matter is given, k is constant, by making $C =$ to $\log. E$, we obtain $p = Ec^{\frac{\phi}{k}}$, since $\because$ p and ρ are functions of ϕ, they will be constant for each stratum of level, but the law of the variation of density is not arbitrary as in the case of incompressible fluids, for the equation $\rho = \frac{p}{k} = \frac{E}{k}.c^{\frac{\phi}{k}}$ determines the law; if the matter of the fluid remaining homogeneous the tem-

perature should vary, k will be a function of the variable temperature, but in order that $\frac{\delta\phi}{k}=\frac{\delta p}{k}$ should be an exact variation, k and consequently the temperature should be functions of ϕ; these functions are arbitrary, hence when the fluid is in a state of equilibrium, the temperature is arbitrary though uniform for each stratum; if this law was given we could integrate $\frac{\delta\phi}{k}$, from which we could infer the law of the densities and pressures by means of the equations

$$p = \mathrm{E}\, e^{\int\frac{\delta\phi}{k}};\ \rho = \frac{\mathrm{E}}{k}.e^{\int\frac{\delta\phi}{k}}\ \&c.$$

(*e*) Let the horizontal surface of the quiescent fluid be the plane of the coordinates of x, y, the axis of z is in this case vertical, which is also the direction of g the accelerating force of gravity; hence $x\,y$ are $= o$, and $\mathrm{R} = g$ in the equation given in page 452, and then δp becomes $= \rho.\, g\delta z$, $\because p = \rho g z$, since $p = o$ when at the surface of the water where $z = o$, there is no constant; calling h the height of any level above the pressed surface, and A the area of the pressed surface, since all the points are equally pressed, and the pressure on each unit of the surface $= p$, Π the pressure on the entire base $= \mathrm{A}.p = \rho.gh\ \mathrm{A}$, $h\mathrm{A} =$ the volume of a cylinder whose base $=$ A and height that of the level of the water, and $\rho gh.$ A is the weight of a corresponding cylinder of water, $\because$ whatever be the shape of the vessel, provided the base and height of the water above the base remain the same, the pressure which the base experiences from the incumbent fluid, remains the same, we suppose in this case that the fluid is in a vacuo, otherwise p does not vanish when $z = o$, and we must have at the surface $p' = \rho.ga$, this is the pressure due to the atmosphere, or to any force which acts equally on all the points of the horizontal surface.

The pressure on each point $\rho g z . \pi = p \pi$, π expressing one of the equal elements of the base, into which the pressed surface is divided, and as the pressure of all the elements are parallel to each other, their resultant is obtained by taking the integral of $\rho g z . \pi$ extended to the entire area, this integral is $=$ to $A z_{,}$, A denoting the area of the pressed surface, and $z_{,}$ the distance of the centre of gravity of this surface from the plane of the level of the fluid; from this it appears that the pressure depends only on the extent of the pressed surface, and on the depth of its centre of gravity below the level of the fluid, therefore if this surface was supposed to revolve about its centre of gravity, the pressure which it experiences will remain the same.

It is easy to estimate the lateral pressure of a fluid in a vessel whose sides are perpendicular to the base, for as the pressure is propagated equally in every direction, the pressure of each molecule is $\div\!\!\div$l to its distance from the horizontal surface of level, hence it is easy to shew that the entire lateral pressure in such a vessel is equal to the weight of a triangular prism of water, whose altitude is that of the fluid, and whose base is a parallelogram, one side of which is equal to the altitude of the vessel, and the other side to its perimeter.

What precedes suggests a method of finding the centre of pressure of a fluid, this centre is that point to which if a force equal to the whole pressure were applied, but in a contrary direction, it would keep the surface at rest, it is therefore the point where the resultant of the pressures of all the elements of the surface meets it, and as the pressures of the elements are parallel forces, the point of application of this resultant must be determined by the theory of the moments of these forces, ∵ as $\rho g z \pi$ denotes the pressure for each element, $\int \rho g z^2 \pi$ expresses the sum of the moments of these elements with respect to the surface of the fluid, which is consequently $= A\, z_{,} z_{,,}$, $z_{,,}$ being the dis-

tance of the centre of pressure from the surface, $\therefore z_{\prime\prime} = \frac{\int \rho g z^{2} \pi}{A z_{\prime}}$, which shews that this centre coincides with that of percussion, hence if a plane surface which is pressed by a liquid be produced to the surface of the liquid, and their common intersection be made the axis of suspension the centre of percussion will be the centre of pressure :—*see* Note (g), Chapter V. This centre of pressure always lower than he centre of gravity except all the points of the surface are equally pressed, in which case they coincide.

(g) Let, as in page 362, P represent the weight of a body in a vacuo, P' its weight in any fluid, V its volume, D its density, Π the weight of the displaced fluid, ρ its density, and g the accelerating force of gravity, we have $P = V.Dg$, $\Pi = V\rho G$, $P - \Pi = P'$, eliminating V and Π, we obtain $\frac{P}{P-P'} = \frac{D}{\rho}$; which equation gives the specific gravity of the body with respect to the specific gravity of the fluid it follows from what is stated in the text, that two bodies which balance in air, are not necessarily of equal weight, unless they are constituted of the same materials; it follows likewise from this, that as $gVD - gV\rho =$ the motive force of a body existing in air, by dividing this expression by V.D, the mass, the quote $= g. \left\{1 - \frac{\rho}{D}\right\}$ (ρ being the density of the air) represents the accelerating force of a body in the air; hence it appears that the air retards the motion of bodies, both because it diminishes its accelerating force, and also because it produces a retarding force depending on the velocity and figure of the moving body. When a body floats on the water, it actually exists in two different fluids, part being in the air and the other part in the water; hence the common rule for determining the specific gravities of

bodies is incorrect; to correct the result we should subtract the number expressing the specific gravity of the air, from the two numbers expressing the specific gravities of the body and fluid on which it floats.

(*h*) A body, whether it floats on a fluid, or whether it is entirely submerged, will be *in equilibrio* when it satisfies the two following conditions, namely, when the centres of gravity of the body and that of the part immersed, or of that of the displaced fluid exist in the same vertical; secondly, when the weight of this portion = that of the body itself; if the body is homogeneous and entirely submerged, the two centres of gravity coincide, and if its density is the same with that of the fluid, it will remain suspended.

(*i*) The body being supposed to be in equilibrio in a fluid, the plane of its intersection with the fluid, which is termed the plane of flotation, will be horizontal, if it then be raised or depressed in a vertical line, and then inclined by an indefinitely small quantity θ to the horizontal position, and a plane parallel to the horizon being supposed to be drawn through the centre of gravity of the first plane, if ζ be the distance of this plane from the present plane of flotation of the fluid, the stability or instability of the fluid depends on the circumstance of $\zeta\ \theta$, which at the commencement are supposed to be very small, remaining always so.

u being the variable velocity of any element dm of the mass of the body, $\int u^2\, dm = C+2\phi$ expresses the sum of the living forces, where ϕ depends on the forces of gravity, and on the vertical pressures which the fluid exerts on all the points of the surface of the body which are submerged under the water; but as the resultant of the motive forces, which are equal for each molecule, to the weight of an equal molecule of the water, is the same as that of the vertical pressures of the fluid, if dv be an element of the volume of the body, corresponding to dm, an element of its mass; the entire motive force

of dm when immersed in the fluid $= gdm - g.\rho.dv$, and therefore from what is established in page 452, $\phi = \int zg\, dm - \int zg\rho.dv$; ∴ if $z_{,}$ represent the distance of the centre of gravity of the entire mass M, from the horizontal plane, we shall have $\int zg.dm = g.\int z.dm = gMz_{,}$; $\int zg\rho dv$ consists of two distinct parts, one relative to the volume V, the part of M which is beneath the original section of the body in its second position, it ∴ $= gV\rho z_{,,}$, $z_{,,}$ being the variable distance of the centre of gravity of V from the plane of flotation; ∴ if K represent the value of $\int zdv$ taken in the limits of V, so that $g.\rho K$ may be the second part of $g.\int z\rho dv$, we shall have $\phi = gMz_{,} - g\rho Vz_{,,} - g\rho K$; but if a be the distance between the centres of gravity of M and V, in the second position of the body, this distance reduced to the vertical $= a.\cos.\theta$, ∴ as the difference between $z_{,}$ and $z_{,,}$ is always $=$ to this reduced distance, we have $z_{,} = z_{,,} \pm a.\cos.\theta$, and ∴ by substituting $\phi = \pm g\rho Va.\cos.\theta - g\rho K$. Now it is not difficult to prove by decomposing the area of the original section into an infinite number of elements, and then projecting them on the plane of flotation, (quantities of the third and higher orders being neglected,) that the value of $K = q.p.$ to $\int zdv = \frac{1}{2}b\zeta.^2 \cos.\theta + \frac{1}{2}\gamma \sin.^2\theta.\cos.\theta$ where $b =$ the area of the original section, and $\gamma = \int l^2 d\lambda$, l being a perpendicular from any point in the original section on the intersection of the original section with the horizontal one drawn through the centre of gravity, and $d\lambda$ an element of the original section; hence we obtain the value of $\phi = \pm g.\rho Va.\cos.\theta - \frac{1}{2}g\rho b\zeta.^2 \cos.\theta - \frac{1}{2}g\rho\gamma \sin.^2\theta.\cos.\theta =$ (neglecting quantities of the third and higher orders) $\pm g\rho Va \mp g\rho Va\theta^2 - \frac{1}{2}g\rho b\zeta^2 - \frac{1}{2}g\rho\gamma.\theta^2$ ∴ $\int u.^2dm + g\rho(\gamma. \pm Va).\theta^2 + g\rho b\zeta^2 = C$; $2g\rho Va$ being contained in the value of C; as θ^2, ζ^2 are positive, it may be shewn as in Notes to page 381 that if $\gamma \pm Va$ is positive, the constant quantity will always remain so; and the value of C de-

pends on the values of $\psi\,\theta\,\zeta$ at the commencement of the motion, it therefore is a very small quantity;

$$\cdot\ \theta^2 \text{ is always } \angle \frac{C}{g\theta(\gamma \pm Va)};\ \zeta \angle \frac{C}{g\rho b};$$

the stability of the equilibrium depends on the sign of $\gamma \pm Va$, and it will be always stable when the coefficient is + at the origin and during the entire duration of the motion; as $\int l^2 d\lambda$ is necessarily +, if Va is also positive, the coefficient $Va + \int l^2 d\lambda$ is +, and the equilibrium is stable, but from what has been established already Va is +, when the centre of gravity of the entire mass is lower than that of the volume of the displaced water; but if this latter centre be the lower, then Va must be taken with a negative sign, and in order that $\gamma - Va$ may be + it is necessary that γ should be $> Va$; but γ varies with the position of the intersection of the horizontal plane with the original plane, which passes through the centre of gravity of the original plane, therefore in its revolution about this centre it must assume different values, and if in that part of the revolution in which γ is a minimum its value predominates over Va, it must do so in all other positions, and $\therefore\ \gamma - Va$ will be always positive; *e.g* in a ship the line, relatively to which γ or $\int l^2 d\lambda$ is a minimum, is evidently the line drawn from the prow to the stern; and if the area of its plane of flotation be divided into an indefinite number of elements, and if the sum of all these multiplied into the square of their respective distances from this line be greater than the product of the volume of displaced water multiplied into the distance of its centre of gravity from that of the vessel, the equilibrium will be stable relatively to all the small oscillations of the vessel.

(*l*) When fluids communicate by means of a level tube, the pressure of each is equal to a cylinder of the fluid whose base A = the common horizontal surface, and whose altitude = the vertical height of the upper surface of the re-

spective fluids above the surface of contact; hence if $s\,s'$ denote the specific gravities of two fluids and $h\,h'$ their respective heights, we have $shA = s'h'A$; hence as we know s', the specific gravity of the air at the earth's surface relatively to s that of the mercury in the barometer the ratio of s to s' gives the ratio of h to h' (the height of the homogemogeneous atmosphere.) It likewise appears that all barometers, whatever may be the diameters of their bores, stand at the same height.

CHAPTER V.

(a) Let the masses of the two bodies A A′ be represented by m and m', their velocities by $v\,v'$, and let u be the common velocity after the shock $\therefore v-u$ will be the velocity lost by A the body, whose velocity is the greater of the two, and $u-v'$ will be the velocity gained by A′; by hypothesis $(m+m')u+m(v-u)+m'.(u-v')$ represents the sum of the quantities of motion previous to the shock, but in consequence of the conditions of equilibrium $m.(v-u)=m'(u-v')$, $\therefore (m+m')u$ is what existed previously to the shock; and it is evident from the preceding equation that the common velocity $u = \dfrac{mv + m'v'}{m+m'}$; if, however A A′ moved in opposite directions with the velocities $v\,v'$, then we would have $m(v-u)=m'(u+v')$ and therefore

$u = \dfrac{mv - m'v'}{m+m'}$, but this value may be comprised in the first

by attending to the signs of the velocities; this effect of the mutual shock of the two bodies is the same as if the forces F F′, which separately actuated m m' to make them acquire the velocities v v', were impressed on m and m' simultaneously, for the velocity communicated by F to $m+m=$ $\frac{mv}{m+m'}$ and that communicated by $F'=\frac{m'v'}{m+m'}$, and $\because$ the velocity of $m+m'$ arising from the combined action of F and $F'=\frac{mv}{m+m'}\pm\frac{m'v'}{m+m'}$, the sign being + or — according as F, F′ act in the same or in contrary directions; if m' has no motion previous to the shock $v'=o$ and $u=\frac{mv}{m+m'}$; $\because$ if m' be very great relatively to m, this quantity vanishes. This is the case with respect to all bodies which impinge on the earth, and all points which are immoveable may be considered as bodies whose masses are infinite relatively to the striking bodies; as in this case, $mv=(m+m')u$, m loses by the shock a quantity of motion $=$ $m'.\ u$, which is that gained by m', *see* Notes, page 440; multiplying the equation $(m+m')\,u=mv\pm m'v'$, by $2u$ and then subtracting from both sides $mv^2+m'v'^2+(m+m')u^2$ we obtain $mv^2+m'v'^2-(m+m')u^2=mv^2+m'v'^2-2u(mv\pm m'v')+(m+m')u^2$ *i. e.* $mv^2+m'v'^2-m'u^2-mu^2=m(v-u)^2+(u\mp v')^2$. $\because$ if the motion of a system of bodies experience a sudden change, there results a diminution in the sum of the living forces of all the bodies = to the sum of the living forces which would arise from the velocities lost or gained by the bodies.—*See* Notes (s) (t) of this Chapter.

(b) In fact if the principle of D. Alembert be applied to determine the circumstances of the impact of two bodies of any form whatever, this principle furnishes us in general with but twelve equations between the unknown quantities of the problem, which in the most general case of it, are thirteen in number, the percussion which the

bodies experience at the instant of the shock being considered as one of them; ∵ there are not a sufficient number of equations to determine these unknown quantities; but the consideration of the compressibility of the two bodies furnishes an additional equation, and thus renders the problem completely determinate.

In order to prove what is asserted in page 279, let, as in the case of non-elastic bodies, v v' be the velocities of m m' previously to impact, they may be assumed respectively $=u+(v-u)$, $u-(u-v')$, let V, V′ be the velocities of m m' after the impact, those being considered as positive which move in the direction of m before the shock, and those as negative which move in an opposite direction, ∵ v will be always positive, but v' u V V′ may be either positive or negative; let u be determined from the equation $m(v-u) = m'(u-v')$, $u-v$, $u-v'$ will be destroyed by the impact, but in consequence of the perfect elasticity of m, m' they will be reflected back with those destroyed velocities; ∵ V the *entire* velocity of m after the shock $= u-(v-u)$ and V′ that of m', $= u+(u-v')$, ∵ substituting for u its value

$$\frac{mv+m'v'}{m+m'}, \text{ we obtain } V=\frac{(m-m')\,v+2m'v'}{m+m'},$$

$$V'=\frac{(m'-m)\,v'+2mv}{m+m'}; \because V-V'=v-v', \text{and if } m=m'; V=v';$$

$V'=v$; we have also by concinnating $mV^2+m'V'^2 = 4u^2(m+m') - 4u(mv+m'v')+m\,v^2+m'v'^2. = mv^2+m'v'^2$.

(c) In general when a body receives an impulsion in any direction, it acquires two different motions, namely, a motion of rotation, and a motion of translation, which are respectively determined by the equations given in page 414, when the three first equations vanish, the forces are reducible to two = and opposite forces acting in parallel directions, when the rotatory motion does not exist the instantaneous forces have an unique resultant passing through the centre

of gravity; when the molecules of the body are solicited by accelerating forces, these in general modify the two motions which have been produced by an initial impulse; if, however, the resultant of the accelerating forces passes through the centre of gravity of the body, the rotatory motion is not affected by them, as is the case with respect to a sphere and planets supposed to be spherical, but in consequence of the oblateness of these bodies the direction of the accelerating force does not pass accurately through the centre; ∵ the axis of rotation does not remain accurately parallel to itself, however the velocity of rotation is not sensibly altered.—*See* Vol. II. Chapter VI.

(*c*) In order to determine the position, &c. of these axes, it is necessary to determine the pressure on a fixed axis which arises from a body revolving about this axis in consequence of a primitive impulse; for this purpose, if this fixed axis be the axis of z, $x\ y\ z$ being the coordinates of dm, $\bar{\omega}$ the angular velocity, and r the distance of dm from the axis of z; the centrifugal force of the element dm is $\div\!\!\cdot$ to $r\bar{\omega}^2$; for it is $=$ to $\frac{r}{T^2}$ and T varying as $\frac{1}{\bar{\omega}}$, it is proportional to $r\bar{\omega}^2$; the fixed axis is therefore urged perpendicularly to its length by the force $r\bar{\omega}^2dm$, and the resultant of all such forces for the sum of the elements dm, or their two resultants, when they are not reducible to one sole force, expresses the entire pressure which the axis experiences during the motion of the body, and as $\frac{x}{r}, \frac{y}{r}$, are the cosines of the angles, which the direction of the force $r\bar{\omega}^2dm$ makes with the axes of x and of y, $x\bar{\omega}^2dm$, $y\bar{\omega}^2.dm$ represent the components of this force resolved parallel to these axes, ∵ $\int x\bar{\omega}^2dm = \bar{\omega}^2\int xdm$ is the resultant of all the forces parallel to the axis of x, which integral is $= \bar{\omega}^2 Mx_{\prime}$, M being the mass of the body, and $x_{\prime}$ the coordinate of the centre of gravity parallel to the axis of x; and $\bar{\omega}^2 My_{\prime}$ ex-

presses the resultant of the forces $\bar{\omega}^2 \int y\, dm$ parallel to the axis of y; if z' z'' represent the respective distances of the resultants $\bar{\omega}^2 Mx_{,}$, $\bar{\omega}^2 My_{,}$, from the plane of x, y, by what is established in Notes page 428, we shall have $Mx_{,}z_{,} = \int xz\, dm$, $My_{,}\, z_{,,} = \int yz\, dm$, by means of these equations we can determine $z_{,}\, z_{,,}$ and the intensities of the forces which acting in the planes xz, yz urge the fixed axis perpendicularly to its length, if $z_{,} = z_{,,}$ the forces $\bar{\omega}^2 My_{,}$ $\bar{\omega}^2 Mx_{,}$, are applied to the same point, and are ∵ reducible to one force, of which the intensity $= \bar{\omega}^2 M(x^2_{,} + y_{,}^2)$ which therefore expresses the pressure on the axis of the body; now if the axis of z be supposed to be *entirely* free, *i. e.* if the body is supposed to revolve in such a manner that the centrifugal forces of the several points do not exercise any pressure on the axis of rotation, and so that this axis has in itself no motion of rotation, neither is it subjected to any pressure; then, not only the moments of the forces which would cause them to revolve about the axis of z, but also the pressures on this axis are = to cypher, *i. e.* $\int xz\, dm = o$, $\int yz\, dm = o$, $\int x\, dm = o$, $\int y\, dm = o$; from the two last it follows, that $x_{,}$ $y_{,}$ are = respectively to cypher, therefore each free axis must pass through the centre of gravity; however, it is evident from the two first, that an axis may pass through the centre of gravity without being free, for $\bar{\omega}^2 \int x dm = \bar{\omega}^2 Mx_{,}$ this quantity $= o$, when the origin is in the centre of gravity; but $z_{,}$ the coordinate of its point of application $= \dfrac{\int x\, z\, dm}{Mx_{,}}$ is infinite, unless $\int x\, z\, dm$ is at the same time equal to cypher.

In order to determine the position of the principal axis of rotation, conceive a plane to pass through this axis perpendicular to the plane xy, and let θ equal the angle formed by the intersection of this plane with this principal axis, and ψ the angle between the axis of x, and this intersection; now, if the position of an element dm be referred to three coordinates x' y' z' of which the first is pa-

rallel to the intersection above mentioned, the second is perpendicular to this intersection, and the third z' is parallel to the axis of z, then it is evident from the transformation of coordinates that we have $x' = x.\cos.\psi + y.\sin.\psi$, $y' = y \cos.\psi - x.\sin.\psi$, $z' = z$, and consequently $x'' = x.\cos.\psi \cos.\theta + y.\sin.\psi.\cos.\theta + z\sin.\theta$; $y'' = z.\cos.\theta - x.\cos.\psi.\sin.\theta - y.\sin.\psi.\sin.\theta$, $z'' = y.\cos.\psi - x.\sin.\psi$, x'', y'', z'' being the coordinates of which the free axis is one; therefore $x''y'' = -\frac{1}{2}x^2.\cos.^2\psi.\sin.2\theta - \frac{1}{2}y^2\sin.^2\psi.\sin 2\theta + \frac{1}{2}z^2\sin.2\theta - xy.\sin.\psi.\cos.\psi.\sin.2\theta + xz\cos.\psi.\cos.2\theta + yz\sin.\psi.\cos.2\theta$; $x''z'' = -x^2.\sin.\psi.\cos.\psi.\cos.\theta + y^2 \sin.\psi.\cos.\psi.\cos.\theta + xy.\cos.\theta.(\cos.^2\psi - \sin.^2\psi) - xz \sin.\psi.\sin.\theta + yz\cos.\psi.\sin.\theta$; ∵ substituting these values in the equation $\frac{\int x''y''.dm}{\cos.2\theta} = o$; $\frac{\int x''z''.dm}{\cos.\theta} = o$; by assuming $\int x^2 dm = A$, $\int y d^2 m = B$, $\int z\, dm = C$; $\int xy dm = D$, $\int xz dm = E$, $\int yz dm = F$, we shall obtain the following equations, $o = -\frac{1}{2}.\tan.2\theta\,(A\cos.^2\psi + B.\sin.^2\psi - C) - D\sin.\psi.\cos.\psi.\tan.2\theta + E.\cos.\psi + F.\sin.\psi.$; $o = (B - A)\sin.\psi.\cos.\psi + D(\cos.^2\psi - \sin.^2\psi) - E\sin.\psi.\tan.\theta + F.\cos.\psi \tan.\theta$; by assuming $\tan.\psi = t$ the last equation gives us

$$\tan.\theta = \frac{(A-B).t + D(t^2-1)}{(F-Et).\sec.\psi};$$

$$\therefore \tan.2\theta = \frac{2(F-Et)\{Dt^2 + (A-B)t - D\}\sec.\psi}{(F-Et)^2.\sec.^2\psi - \{Dt^2 + (A-B)t - D\}^2};$$

but by means of the first equation we obtain

$$\tan.2\theta = \frac{2(E+Ft).\sec.\psi}{(B-C)t^2\ 2Dt + A - C};$$

by equating these two values we derive an equation of the fifth degree and divisible by $\sec.^2\psi = 1 + t^2$; ∵ t will be finally given by an equation of the third degree; therefore for every body, there is either one or three real values of $t = \tan.\psi$. and consequently of ψ, and as

$$\tan.\theta = \frac{(A-B)\sin.2\psi - 2D\cos.2\psi}{2(F\cos.\psi - E\sin.\psi)}$$

it is evident that we have always a real value for θ, and thus by means of the angles ψ and θ we can determine the position of a free axis for every body; but in point of fact, the three roots of the cubic equation are real, and therefore there are three free axes belonging to every body; for if the axis of x be the free axis determined by that one of the three roots which is given to be real, and if the axis of x'' of which the position is determined by ψ, θ, was also free, when these angles are always positive, the three roots are real, and ∵ then besides the axis of x'' the axis of x is also free; x'', y'', z'', x, y, z, representing the same as before, if x be a free axis we have $\int x\, y\, dm = D = 0$, $\int x\, z\, dm = E = 0$; consequently the axis of x' will be also a free axis, if the equations $\int xydm$ $\int xzdm$ respectively $= 0$, after substituting for them $D=0$, $E=0$, give real values for ψ and θ, but these equations then become, $o = F.\sin.\psi - \frac{1}{2}.\tan.2\,\theta.(A\cos.^2\psi + B.\sin.^2\psi - C)$, $o = F.\cos.\psi.\tan.\theta - (A - B)\sin.\psi.\cos.\psi$; these two equations of condition must be satisfied when the axes of x and x' are simultaneously free axes, but the last may be satisfied by supposing $\cos.\psi = 0$, or $\psi = 90°$, and $\sin.\psi = 1$. By substituting this value in the first equation we obtain $\tan.2\theta = \frac{2F}{B-C}$, which is satisfied by θ or $\theta' = \theta + 90$; since $\psi = 90$, and the *planes* xx', $x'x''$ are by hypothesis perpendicular to each other, the angle between the axis of x'' and the axis of x is also $90°$, and therefore the axis of x is perpendicular to the two others, and as $\theta' - \theta = 90$, it follows that every body has at least three axes intersecting each other perpendicularly in their centre of gravity. It is evident from the preceding analysis that the values of θ ψ are deduced by assuming $\int xydm = o$; which it appears from the reality of the roots may obtain without the equations $\int x\, dm = o$, $\int y\, dm = o$, having place, if these equations do not vanish neither $x_{\prime}$ nor $y_{\prime}$ will va-

nish, but as by hypothesis, $\int xzdm$, $\int yzdm=0$ respectively, z' z'' will vanish; and the axis of rotation will experience a pressure represented by $\bar{\omega}^2 M\sqrt{x_{\prime}^2+y_{\prime}^2}$ applied at the origin of the coordinates, *see* page 466; in this case therefore if the origin be fixed, the pressure arising from the centrifugal forces will be destroyed; hence if the axis of rotation for which $\int y\,xdm=o$, $\int z\,xdm=o$, be at liberty to turn about the fixed point, the body will revolve about this axis, as if it was fixed; consequently it appears that a *fixed* point being given in a body of any figure whatever, there always exist three axes passing through this point, about which the body may revolve uniformly, without any displacement of these axes, and just as if these axes were altogether immoveable, and these are the only axes which possess this property, for supposing the body to revolve about any other axis passing through the fixed point, this axis would evidently experience a pressure, which would not pass through the fixed point; since then z' and z'' would no longer vanish, if the axis was suddenly remitted to revolve freely about the fixed point the pressure will no longer be destroyed, therefore this force would displace the axis of rotation and the motion would be deranged; it appears from this that if a body impinge on another retained by one sole point, its motion will be continued uniformly, if it commences to revolve about one of the principal axes which intersect in this fixed point, in the same manner as if the axis were fixed, in this case it is necessary that the percussion on the axis of rotation at the commencement of the motion be reduced to one sole force perpendicular to this axis and passing through the fixed point, for it is then counteracted by the resistance of the fixed point.

This sum of the products of each molecule of the body into the square of its distance from an axis is termed the moment of inertia of the body with respect to that axis.

If r, r', r'', denote the distances of the element dm from the axis of x, y, z, respectively, we have $r = \sqrt{y^2+z^2}$, $r' = \sqrt{x^2+z^2}$, $r'' = \sqrt{x^2+y^2}$ the moments of inertia relatively to x, y, z, are $\int r^2dm = (B+C)$, $\int r'^2dm = (A+C)$, $\int r''^2dm = A+B$ respectively, which are severally $= Ma^2$, Mb^2, Mc^2, a, b, c, being the coordinates of the centre of gravity, and besides these three equations between A, B, C, we have also $D=o$, $E=o$, $F=o$; x, y, z being free axes; relatively to any other axis x', of which the position, with respect to the plane xy, is determined by θ and ψ, we have the moment of inertia $x'' = \int(y''^2+z'^2)dm = A.(\sin.^2\psi+\cos.^2\psi \sin.^2\theta)+B(\cos.^2\psi+\sin.^2\psi.\sin.^2\theta)+C\cos.^2\theta = Mk^2$; for each axis situated in the plane of xy, we have $\theta = o$, and the moment of inertia with respect to such an axis $= A \sin.^2\psi + B.\cos.^2\psi + C = Mm^2$, $\because Ma^2 - Mm^2 = (B-A).\sin.^2\psi$, $Mb^2 - Mm^2 = (A-B).\cos.^2\psi$, $Mc^2 - Mm^2 = (A\cos.^2\psi + B\sin.^2\psi - C$, $Mk^2 - Mm^2 = (A\cos.^2\psi + B\sin.^2\psi - C).\sin.^2\theta$; whatever, therefore, be the value of ψ, or the *position* of the plane passing through the axis of x at right angles to the plane x, y, the difference between the moments of inertia relatively to a line which coincides with the intersection of these planes and to any axis x'' existing in the perpendicular plane $(= Mk^2 - Mm^2,)$ will be a *maximum*, when $\theta = o$, in which case the axis of x'' coincides with z, a free axis of rotation, *i. e.* the difference between the moments of inertia with respect to the free axis of z, and an axis perpendicular to it existing in the plane of x, y which is also a free axis, is $>$ than the difference between the moments of inertia relatively to the first of these axes and any other axis, therefore the moment of inertia relatively to the first axis is $>$ or $<$ than with respect to any other axis according as it is $>$ or $\angle$ than relatively to the axis in the plane $x\,y$, $\because$ the moment of inertia with respect to the the first axis is either a *maximum* or *minimum*. As the same may be shewn to be true with respect to the other two principal axes at right angles to this first axis, it fol-

lows in general that in every body, the moment of inertia with respect to each of the three free axes is a maximum or minimum. But as from the nature of moments of inertia in general, they can neither become negative or infinite, it follows, that these three moments are alternately maxima or minima. If $\psi = o$, the intersection of the two planes coincides with the axis of x, and the moment with respect to the axis of x'' being by what goes before $>$ or $\angle$ than that relatively to the axis of x, the moment relatively to the first axis will be so of course, $\because$ if the moments relatively to the axis of x and the first axis are $=$, the moments relatively to all axes existing in the plane of these axes will necessarily be $=$; this is equally true for the other two pair of axes. In every case one of the three free axes may be considered as that to which the greatest moment appertains, and the other as that to which the least appertains, so that $Ma^2 > Mb^2$ and $Mb^2 > Mc^2$, $\because$ if $a^2 = c^2 + \alpha^2$, $b^2 = c^2 + \beta^2$; $c^2 = a^2 - \alpha^2$, $b^2 = a^2 - \gamma^2$, $\alpha\ \beta\ \gamma$ vanish if two or three of the moments are equal; from what precedes $A = \frac{1}{2}M(b^2 + c^2 - a^2)$, $B = \frac{1}{2}M(a^2 + c^2 - b^2)$, $C = \frac{1}{2}M(a^2 + b^2 - c^2)$, this being substituted in the value of $\int(y'^2 + x''^2).dm$, given in preceding page, will give $Mk^2 = Ma^2.\cos.^2\psi.\ \cos.^2\theta + Mb^2.\sin.^2\psi\cos.^2\theta. + Mc^2.\sin.^2\theta. =$ by substituting for $a^2\ b^2$ their values $c^2 + \alpha^2$, $c^2 + \beta^2$, $Mc^2 + M\alpha^2.\cos.^2\psi.\cos.^2\theta + M\beta^2\sin.^2\psi.\cos.^2\theta$, (or by substituting $a^2 - \gamma^2$ for b^2, and $a^2 - \alpha^2$ for c^2) $= Ma^2 - M\gamma^2.\sin.^2\psi.\cos.^2\theta - M\alpha^2.\sin.^2\theta$, $\because$ all moments Mk^2 are less than the greatest and greater than the least of those which belong to the three free axes, one of these last is an absolute maximum, and the other an absolute minimum. If Ma^2, Mb^2 Mc^2 the three principal moments are equal; α, β, γ are $=$ to cypher, and $Mk^2 = Ma^2 = Mc^2 = Mb^2$, $\because$ all the moments of the body are $=$. If only two of the moments Ma^2 and Mb^2 are equal, $\gamma = o$, and $Mk^2 = Ma^2 - M\alpha^2.\sin.^2\theta = Ma^2.\cos.^2\theta + Mc^2.\sin.^2\theta$, and by making $\theta = o$, $Mk^2 =$

$Ma^2 = Mb^2$, $\therefore$ all moments relatively to axes in the plane of $x\,y$ are $=$, which is indeed evident of itself, for if Mk^2 was $>$ or $\angle$ Ma^2, the difference between Ma^2 and Mb^2 would be still greater. It is on account of these remarkable properties that these three axes have been termed principle axes, since Mk^2 is $\angle M.a^2$ and $> Mc^2$, $k \angle a$ and $> c$, $\therefore b \angle a$ and $> c$, if $B = C$, then $Mb^2 = A + C = Mc^2 = A + B$, $\therefore$ if $\psi = o$, we shall have because $D = E = F = o$, $\tan. 2\theta = \frac{0}{0}$; therefore the angle θ is indeterminate, $\therefore$ all axes which exist in the plane x, y, satisfy the conditions of free axes, and are therefore principal axes; if $A = B = C$ the three moments are then equal, namely, $Ma^2 = Mb^2 = Mc^2$, $\therefore$ sin. $\psi = \frac{0}{0}$ *i. e.* the two angles θ and ψ are indeterminate.

(*f*) If the body is not actuated by any extraneous forces, the equations of its motion of rotation are

$$\frac{dp}{dt} + \frac{C^2 - B^2}{C^2}.\ qr. = 0; \quad \frac{dq}{dt} + \frac{A^2 - C^2}{B^2}.pr = 0,$$

$$\frac{dr}{dt} + \frac{B^2 - A^2}{C^2}.pq = 0.$$

where $p = \bar{\omega}.\ \cos.\ \alpha$, $q = \bar{\omega}.\ \cos.\ \beta$, $r = \bar{\omega}.\ \cos.\ \gamma$, $\cos.\ \alpha$ being supposed $= \cos.\ \psi.\ \cos.\ \theta$, $\cos.\ \beta = \cos.\ \psi.\ \sin.\ \theta$, $\cos.\ \gamma = \sin.\ \theta$, and ψ, θ denote the same as in page 467, and as $\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1$, we have $\bar{\omega}$ the angular velocity $= \sqrt{p^2 + q^2 + r^2}$. (Celestial Mechanics, p. 207). In order that the motion should be uniform about an invariable axis, it is necessary that the quantities p, q, r, should be constant; hence we have the three following equations of condition:

$$(C^2 - B^2).\ qr = 0.\ (A^2 - C^2).\ pr = 0.\ (B^2 - A^2).\ pq = 0;$$

we satisfy these conditions by assuming $A = B = C$, *i. e.* if all the moments of inertia are equal, and consequently all the diameters of the revolving body principal axes, the simplest case of this is that of an homogeneous sphere; it is also satisfied if two of the quantities p, q, r, vanish,

which is evidently the case when the body revolves about a principal axis; in fact, if it be the axis of z, then $\alpha = \beta = 90$, and $\gamma = 0$; hence, it follows, that $p = 0$, $q = 0$, and $r = \bar{\omega}$. The rotation is therefore uniform and invariable if the solid revolves about a principal axis, and conversely, the rotation cannot be uniform except about a principal axis; in fact, either the three moments of inertia A, B, C, are unequal, or only two of them A, B, are $=$, or all the three are equal; in the first case, pq, pr, qr, and therefore two of the quantities p, q, r, must vanish, or which is the same thing, two of the angles α, β, γ must be right angles, and the third $= 0$, in order that the motion may be uniform; hence it follows, that the axis of rotation is a principal axis; in the second case $pr = qr$, therefore $p = q = 0$, or $r = 0$, the first supposition makes $\alpha = \beta = 90$, and $\gamma = 0$, because $\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 0$, therefore the axis of rotation is a principal axis.

The second supposition $r = 0$, gives $\gamma = 90$, therefore the axis of rotation coincides with the plane of x, y, in which all the diameters are principal axes, because the two moments A, B, are equal. In the third case, $A = B = C$, the quantities p, q, r are undetermined, and as all diameters which pass through the centre of gravity are in this case principal axes, the axis of rotation will be one also. In all cases in which the axis of rotation is not a principal axis, whether the solid be free or solicited by extraneous forces, the velocity of the rotation as well as the position of the axis will be liable to changes, which depend on the conformation of the solid, *i. e.* on the quantities A, B, C, and on the position of the axis of rotation with respect to the principal axes, *i. e.* on p, q, r. If the axis of rotation is inclined to the principal axis in a very small angle, this axis being that of z, the quantities p, p are so very small, that we may neglect their product, therefore $dr = 0$, and $\because r = h$ a constant quantity. In

this case also the velocity of rotation $= \omega$ is nearly constant; hence the two other equations become $0 = dp + \frac{C^2-B^2}{A^2}. hq.dt$, $0 = dq + \frac{A^2-C^2}{B^2}. hp.dt$; and it is evident that the integral of these equations must assume the form $p = m. \sin.(nt+e)$, $q = m' \cos. (nt+e)$, n, m', n, e being constant quantities, it is easy to prove by substitution that these circular functions satisfy the preceding differential equations, and therefore may be assumed as the values of p and q. See Celestial Mechanics, page 208. We may deduce from them $0 = mn + \frac{C^2-B^2}{A^2}. hm'$; $0 = -m'n + \frac{A^2-C^2}{B^2}. hm$, hence we have $n = \frac{B}{AB}. \sqrt{(A^2-C^2).(B^2-C^2)}$, and $m' = \frac{A}{B}. m \sqrt{\frac{A^2-C^2}{B^2-C^2}}$. If $(A^2 - C^2). (B^2 - C^2)$ is positive, *i. e.* if the moment of inertia with respect to the axis of rotation Mc^2, is greater or less than Ma^2 and Mb^2, and consequently the greatest or least of all the moments, n, m, m' are real quantities, and p, q are expressed by real sines, therefore the variations of rotation being periodic, and confined within very narrow limits, the axis of rotation will make small oscillations about its primitive state, the magnitude of which may be determined by the equations $p = m. \sin. e$, $q = m'. \cos. e$. As by hypothesis, p, q, were at the commencement of the motion extremely small, m, m' are extremely small, and thus p, q will always differ very little from cypher. The state of rotation is therefore stable, if the body commenced to move about an axis inclined in a very small angle to one of the two principal axes, of which the moments of inertia are the greatest or least of all; the velocity will then experience only insensible and periodic oscillations, and the axis of rotation will make slight excursions about the principal axis, the rotation always returning to its primitive state; but if the

solid revolves nearly about the principal axis, of which the moment of inertia Mc^2 exists between the two others, Ma^2 and Mb^2, the rotation will be subject to changes, which instead of being periodic may increase indefinitely; for n being then imaginary, the sine and cosine of $nt+e$, will be changed into exponentials, which are susceptible of continual increase; in this case, therefore, the motion of rotation is not stable, and the slightest derangement may cause the changes to be indefinitely great. And as observation proves that the rotation of the sun, planets, and satellites, (which are observed) is in a stable state, it appears certain that all the celestial bodies revolve very nearly about a principal axis, with respect to which the moment of inertia is the greatest or least, most probably the first, for on account of the compression of the earth arising from the rotation, the axis is smaller than the diameter of the equator, and therefore its moment of inertia is greater. *See* Tom. II. Chap. VI.

(g) Suppose the axis of x' to be this horizontal axis, and if the axis of y' be also horizontal, the axis of z' will be vertical, let the plane which passes through the axis of y' and z', pass through the centre of gravity, and let ϕ be the angle which the axis of z' makes with an axis passing through the centre of gravity and the origin of the coordinates; if y'', z'' be the coordinates referred to this new axis, then $y'=y''.\cos.\phi+z''.\sin,\phi$; $z'=z''.\cos.\phi-y''.\sin.\phi$; now as the coordinates y'', z'' are constantly the same for the same body, and only vary in passing from one molecule to another, by taking the differential of y' and z' with respect to the time, we obtain

$$f.\frac{(x'dy'-y'dx')}{dt}.\ dm=-\frac{d\theta}{dt}.\ f.\ dm.\ (y^{2\prime\prime}+z^{2\prime\prime}),$$

$f.dm.(y^{2\prime\prime}+z^{2\prime\prime})$ is the moment of inertia of the body with respect to the axis of x'; if this moment $=C'$, then from what is stated in page 443, multiplying by dm, and ex-

tending the expression to all the molecules, we obtain $\int.\frac{y'.dz'-z'.dy'}{dt}.dm = V = -C'.\frac{d\theta}{dt}$, and as C' is constant we have $-C'\frac{d^2\theta}{dt^2}=\frac{dV}{dt}$, if the only force actuating the body be that of gravity, then the values of P, Q, which are supposed to act horizontally, will vanish, and R, which acts vertically, will be constant; hence, we obtain

$$\frac{dV}{dt}=\int.\mathrm{R}y'dm= \mathrm{R}.\cos.\theta.\int.y''.dm+\mathrm{R}.\sin.\theta\int z''.dm,$$

since the axis of z'' passes through the centre of gravity of the body, $\int y''.dm = 0$, and if h be the distance of the centre of gravity of the body from the axis of x', $\int.z''.dm=\mathrm{M}h$, M being the entire mass of the body, therefore $\frac{dV}{dt}=\mathrm{M}h.\ \mathrm{R}.\ \sin.\ \theta$, and consequently $\frac{d^2\theta}{dt}$

$$= -\frac{\mathrm{M}h.\ \mathrm{R}.\ \sin.\ \theta}{C'};$$

suppose a second body, all whose molecules are condensed into one point, of which the distance from the axis of x' is $=$ to l, we shall have for this body $C'=\mathrm{M}'l^2$, M' expressing the mass, for as all the molecules are condensed into one point

$$h = l, \text{ and } \frac{d^2\theta}{dt^2}=-\frac{\mathrm{M}'l}{\mathrm{M}'l^2}.\ \mathrm{R}.\ \sin.\ \theta = -\frac{\mathrm{R}}{l}.\ \sin.\ \theta,$$

moreover $\frac{d^2\theta}{dt^2}=-\frac{\mathrm{R}}{l}.\ \sin.\ \theta$, hence the two bodies will have the same oscillatory motion, if their initial angular velocities are the same, when their centre of gravity exists in the same vertical, and when $l=\frac{C'}{\mathrm{M}h}$, which is equivalent to the rule given in the text. Multiplying both sides of the equation $\frac{d^2\theta}{dt^2}=-\frac{\mathrm{R}.\sin.\theta}{l}$ by $2d\theta$, then

integrating we obtain $\frac{d\theta^2}{dt^2}=\frac{2R}{l}.\cos.\theta+B'$, the constant quantity B′ depends on the angular velocity and value of θ at the commencement of the motion. From the equation $l=\frac{C'}{Mh}$, it appears that when the axis of rotation passes through the centre of gravity, $h=0$, and l is infinite, therefore the time of oscillation is infinite; in fact, in this case the action of gravity being destroyed the primitive impulse will communicate a motion of rotation, which will be perpetuated for ever if the resistance of extraneous causes be removed. The point which is distant from the axis of rotation by a quantity equal to l is termed the centre of oscillation of the body; and if the axis of rotation passed through this point, the centre of oscillation, with respect to the new axis, will be in the former axis of rotation; for the moment of inertia, with respect to the centre of gravity, being equal to $C'-Mh^2$, the moment of inertia with respect to the new axis will be $C'+Ml^2-2Mlh$, therefore the value of l for the new axis $=\frac{C+Ml^2-2Mlh}{Ml-Mh}$, but $C'=Mlh$, therefore the value of l for the new axis $=\frac{Ml^2-Mlh}{Ml-Mh}=l.$

Let $C'=A.\sin.^2\theta.\sin.^2\phi+B.\sin.^2\theta.\cos.^2\phi+C.\cos.^2\theta+Mh^2$, A, B, C, being the moments of inertia relatively to the principal axes passing through the centre of gravity, see page 467, we shall have $l=$

$$\frac{Mh^2+A.\sin.^2\theta.\sin.^2\phi+B.\sin.^2\theta.\cos.^2\phi+C.\cos.^2\theta}{Mh},$$

therefore l will be a minimum when the quantity C′ becomes the least of the three principal moments of inertia, for in that case the two other moments must vanish; let A be the least of these moments, then $l=\frac{Mh^2+A}{Mh}$, for

sin. θ. cos. $\phi = 0$, cos. $\theta = 0$, and to determine when l is a minimum, $d\theta = \frac{2M^2.h^2 - M^2h^2 - MA}{M^2h^2}$. $dh = 0$, hence $h = \sqrt{\frac{A}{M}}$, therefore l, and consequently the time of rotation will be a minimum when the axis of rotation is that principal axis relatively to which the moment of inertia is a minimum, and at a distance from the centre of gravity by a quantity $= \sqrt{\frac{A}{M}}$; lh is constant and $= \frac{C'}{M}$, which is equal to the square of the distance of a point called the centre of gyration from the axis of rotation, *i. e.* that point where if all the matter contained in the revolving body was collected, any point to which a given force is applied to communicate motion would be accelerated in the same manner as when the parts of the system revolve in their respective places, and consequently the same angular velocity is generated in both cases; therefore this distance is a geometric mean proportion between the distances of the centres of gravity and oscillation from the axis of rotation, and from what precedes it appears that when the time of vibration is a *minimum*, the distance of the centre of gyration from the axis of rotation is equal to the distance of the centre of gravity from the same, and the distance of the centre of oscillation from the same axis $= 2.\sqrt{\frac{A}{M}}$, in this case the centre of gyration is termed the principal centre of gyration.

If, as in the case of the planets the rotatory motion arises from a primitive impulse, of which the direction does not pass through the centre of gravity, then, in consequence of what is stated in notes (c) and (u), it follows this centre will move in the same manner as if the impulse was applied immediately to it, and the rotatory motion about this centre will be the same as if it was fixed; the sum of

the areas described about this point by the radius vector of each molecule projected on the plane passing through the centre of gravity, and the direction of the impulsion multiplied respectively by these molecules will be proportional to the moment of the primitive force projected on the same plane; but this moment is evidently the greatest possible, for the plane which passes through its direction and through the centre of gravity, therefore this is the invariable plane. (*see* note (x) of this chapter.) If f be the distance of the primitive impulse from the centre of gravity, and v the velocity impressed on this point, Mfv will be the moment of the impulsion, and being multiplied by $\frac{1}{2}t$, the product will be equal to the sum of the areas described in t; but, as will be seen hereafter, this sum is equal to

$$\sqrt{C^2p^2+A^2q^2+B^2r^2},$$

$$\therefore mfv=\sqrt{C^2p^2+A^2q^2+B^2r^2},$$

and if at the commencement of the motion we know the position of the principal axis with respect to the invariable plane, *i. e.* the angles θ and ϕ, we shall have the values Cp, Aq, Br, at the commencement, and therefore at any subsequent instant. Now, if the moving body was a sphere, of which the radius $=R$, and if U be the angular velocity with which it revolves about the sun, the distance from the sun being $=d$, $v=dU$, and as the planet is put in motion by a primitive impulse, the axis of rotation will be perpendicular to the invariable plane; and on the hypothesis that this axis coincides with the third principal axis, $\theta=0$, therefore $Aq=0$, $Br=0$; consequently $Cp=mfv=mfdU$; in a sphere $C=\frac{2}{5}\cdot\ mR^2$,

therefore $f=\frac{2}{5}\cdot\frac{R^2}{d}\cdot\frac{p}{U}$, by means of which we can determine the distance of the direction of the force which causes a planet to revolve about the sun with a velocity

of rotation $=p$, and a velocity of revolution $=U$. (*See* note (*c*).

(*h*) If a body describes an ellipse, the centre of the ellipse being the point to which the force is directed, the force will vary as the distance from the centre, and *vice versa* if the force vary as the distance, the curve describes an ellipse, the point to which the force is directed being in the centre, but it is evident that in cases of small impulsions made on the vibrating body, the force varies very nearly at the distance. The time of the revolution is twice the duration of the vibration of a pendulum whose length is the distance of the plane of the ellipse described from the point of suspension.

The general solution of the problem of the very small oscillations of a system of bodies about their points of equilibrium, is very complicated. However, the following may be considered as a *precis* of the method of Lagrange: he assumes that the coordinates of the several bodies may be expressed by the coordinates which appertain to the body in a state of equilibrium, increased by the very small variables which vanish in the state of equilibrium; this is always possible when the equations of condition, reduced into a series, contain the first powers of the variables, which are assumed to be extremely small; as for instance, if a, b, c, be the coordinates of a body in a state of equilibrium, when it deviates very little from this state, let the coordinate $x=a+\alpha$, $y=b+\beta$, $z=c+\gamma$, α, β, γ, being so extremely small that powers of them higher than the first may be neglected; then if the equations of condition $L=0$, $M=0$, &c., are in any position algebraic functions of x, y, z, x', &c.; as the position of equilibrium is one of the positions of the system, it is evident that the equations $L=0$, $M=0$, &c. must still subsist; x, y, z, x', &c. being supposed to become a, b, c, a', &c. hence, it is evident that these equations do not involve the time t, let A, B, &c. be what L, M, &c. become when x, y, z, x',

become a, b, c, a', &c. it is evident that by substituting for x, y, z, x', &c. their values $a+\alpha$, $b+\beta$, $c+\gamma$, &c.

$$L = A + \frac{dA}{da}.\alpha + \frac{dA}{db}.\beta + \frac{dA}{dc}.\gamma + \frac{dA}{da'}.\alpha' + \text{\&c.};$$

$$M = B + \frac{dB}{da}.\alpha + \frac{dB}{db}.\beta + \frac{dB}{dc}.\gamma + \frac{d\beta}{da'}.\alpha', \text{\&c.}$$

∵ as relatively to the state of equilibrium $A = 0$, $B = 0$, The values $L-A$, $M-B$, are respectively equal to cypher, which will give the relation which ought to subsist between α, β, γ, α', &c. and by neglecting very small quantities of the second and higher orders, we will obtain linear equations by means of which we can determine the values of some of these variables in terms of the others, then by means of these first values, we shall find others more exact, taking into account the second, and even higher powers, as we wish.

(k) In general assuming $x = a + a_{,}\xi + a_{,,}\psi + a_{,,,}\phi$, &c. $y = b + b_{,}\xi + b_{,,}\psi + b_{,,,}\phi$, &c. $z = c + c_{,}\xi + c_{,,}\psi + b_{,,,}\phi$ +&c., where a, $a_{,}'$, $a_{,,}$, &c. b, $b_{,}$, $b_{,,}$, &c., c, $c_{,}$, $c_{,,,}$, &c. are constant, and ξ, ψ, ϕ, &c. are very small variable quantities, which are = to cypher in the case of equilibrium; when the variables ξ, ψ, ϕ, &c. are supposed to be in a constant ratio to each other, then in the expression for the sum of the living forces, and for its variation, we would arrive at an equation of the form $\frac{d^2\zeta}{dt^2} + k\xi = 0$, of which the integral is $\zeta = E. \sin. t.\sqrt{k+e}$, where k has as many values as there are unknown quantities; it is evident, that this expression represents the very small isochronous oscillations of a simple pendulum, the length of which is equal to $\frac{g}{k}$, g representing the force of gravity, therefore the oscillations of the different bodies of the system may be considered as made up of small oscillations

analogous to those of pendulums, the lengths of which are $\frac{g}{k}, \frac{g}{k'}, \frac{g}{k''}$.

(m) As the coefficients E, E′, E″, &c. depend solely on the initial state of the system, we may always suppose this state to be such that all the coefficients E′, E″, E‴, &c. except one vanishes, then all the bodies of the system make simple oscillations analogous to those of the same pendulum, and it thus appears that the same system is susceptible of as many different simple oscillations, as there are moveable bodies; therefore, generally speaking, the oscillations of the system, of what kind soever they are, will only be made up of those simple oscillations, which from the nature of the system may have place; consequently however irregular the small oscillations which are observed in nature appear to be, they may be always reduced to simple oscillations, the number of which is equal to the number of vibrating bodies in the same system; this immediately follows from the linear equations, by which the motions of the body which compose any system are expressed, when those motions are very small. The system can never resume its original position when $\sqrt{k'}$, $\sqrt{k''}$, $\sqrt{k'''}$, &c. are incommensurable, for in that case the times of the oscillations are incommensurable:—if they are commensurate the system will return to the same position at the end of the time $T = \frac{2\pi}{\mu}$, where $\pi = 180$, and $\mu =$ the greatest common measure of $\sqrt{k'}$, $\sqrt{k''}$, $\sqrt{k'''}$, &c. θ will then be equal to the time of the compound oscillation of the system.

(n) This principle is called the principle of D'Alembert, as it was first announced by that philosopher, by means of it the laws of the motion of a system are reducible to one sole principle, in the same manner as the laws of the equilibrium of bodies have been reduced to the principle of virtual velocities.

In consequence of the mutual connexion which subsists between the several bodies of the system, the effect which the forces immediately applied to the several parts of the system would produce, is modified so that their velocities, and the directions of their motions are different from what would take place if the bodies composing the system were altogether free; therefore if at any instant if we compute the motions which the bodies would have at the subsequent instant, if they were not subject to their mutual action, and if we also compute the motions which they have, in the subsequent instant, in consequence of their mutual actions, the motions which must be compounded with the first of these in order to produce the second, are such, as if they acted on the system alone, would constitute an equilibrium between the bodies of the system, for if not, the second of the above-mentioned motions are not those which have actually place, contrary to hypothesis, *i. e.* if v, v', v'', &c. be the velocities which the bodies m, m', m'', &c. composing the system would have if each of them was isolated, and if u, u', u'', &c. are the unknown velocities with which the bodies are actuated in directions equally unknown, in consequence of the mutual connexion of the parts of the system; and if p, p', p'', be the velocities which must be compounded with u, u', u'', &c. acting in a contrary direction, in order to produce v, v', v'', &c. respectively, then there is evidently an equilibrium between mp, $m'p'$, $m''p''$, &c., the quantities of motion lost or gained; otherwise, u, u', u'', would not be the velocities which have actually place; as mp is the resultant of mu and of mv, taken in a direction the contrary to its motion, by substituting for mp, $m'p'$, $m''p''$, &c. their components, we may announce the principle by stating that there is an equilibrium in the system between the quantities of motion mv, $m'v'$, $m''v''$, &c. impressed on the bodies, and the quantities mu, $m'u'$, $m''u''$,

&c. which actually obtain; these latter being taken in a contrary direction, by announcing the principle in this way we avoid complicated and embarrassing resolutions, and we need not consider the quantities of motion lost or gained, besides we are enabled by it to establish directly equations of equilibrium between the *given* velocities v, v', v'', &c., and the unknown velocities u, u', u'', &c., which can therefore be determined by means of these equations. However it must be observed, that the above equation is not sufficient of itself to determine u, u', u'', &c. we must in addition, obtain another to be determined by the nature of the system.

If the bodies are actuated by accelerating forces, then if those resolved parallel to the coordinates x, y, z, be P, Q, R, for m, P′, Q′, R′, for m', &c., mP, mQ, mR, mP′, &c., will represent the motions parallel to the three axes which the bodies would have if they were altogether free, and

$$m.\frac{d^2x}{dt^2},\ m.\frac{d^2y}{dt^2},\ m.\frac{d^2z}{dt^2},$$

represent the motions parallel to the same, which the bodies actually have at the commencement of the second instant, which since they are to be taken in a direction opposite to their true one, must be affected with contrary signs to mP, mQ, mR. *See* page 432.

$$\because -m.\left(d.\frac{dx}{dt}+\text{P}.dt\right);\quad -m.\left(d.\frac{dy}{dt}+\text{Q}.dt\right);$$

$$-m.\left(d.\frac{dz}{dt}+\text{R}.dt\right),\ \&\text{c.}$$

will be destroyed.

$$i.\ e.\ \text{if}\ m.\frac{dz}{dt},\ m.\frac{dy}{dt},\ m.\frac{dz}{dt},$$

represent the partial forces of the body m at any instant, resolved parallel to the three axes, in the subsequent instant they will become

$$m.\frac{dx}{dt}+m.d.\frac{dx}{dt}-m.d.\frac{dx}{dt}+m.\,Pdt.$$

$$m.\frac{dy}{dt}+m.d.\frac{dy}{dt}-m.d.\frac{dy}{dt}+m.\,Q.dt+\text{\&c.}$$

and as $m.\frac{dx}{dt}+m.d.\frac{dx}{dt},\ m.\frac{dy}{dt}+m.d.\frac{dy}{dt}$, &c.

only remain in the subsequent instant,

$$-m.d.\frac{dx}{dt}+m.P.dt,\ -m.d.\frac{dy}{dt}+m.\,Q.dt+\text{\&c.}$$

will be destroyed; by distinguishing in this expression the characters in x, y, z, P, Q, R, by one, two, &c., marks we shall have an expression for the velocities destroyed in m', m'', &c., and multiplying these forces by $\delta x, \delta y, \delta z$, &c., the respective variations of their directions, by means of the principle of virtual velocities, the following equation will be obtained,

$$0=m\delta x.\left(\frac{d^2x}{dt}-P.\right)+m.\delta y.\left(\frac{d^2y}{dt^2}--Q\right)$$

$$+m.\delta z.\left(\frac{d^2z}{dt^2}-R\right)+m'\delta x'.\left(\frac{d^2x'}{dt^2}-P'\right)+\text{\&c.}$$

if we eliminate by means of the particular conditions of the parts of the system as many variations as there are conditions, and then make the coefficients of the remaining variations separately equal to cypher, we shall obtain all the equations necessary for determining the motions of the bodies of the system.

As $\frac{d^2x}{dt^2}$ is made to express the increase of the velocity, the changes in the motion of m are made by insensible degrees. The preceding equation consists of two parts, entirely distinct, namely,

$$\Sigma.m.(P.\delta x+Q.\delta y+R.\delta z),$$

$$\text{and }\Sigma m\left(\frac{d^2x}{dt^2}.\,\delta x+\frac{d^2y}{dt^2}.\,\delta y+\frac{d^2z}{dt^2}.\,\delta z\right),\ \text{\&c.}$$

the first member would be equal to cypher if P, Q, R, P′, &c., which are applied to the several bodies of the system constituted an equilibrium; the other part arises from the motion which is produced by the forces P, Q, R, P′, &c., when they do not constitute an equilibrium, and the equation in page 431 is only a particular case of this; the second member is totally independent of the position of the axis of the coordinates, for substituting

$$\text{for } x, \ \ ax' + by' + cz', \ \text{for } y, \ \ a'x' + b'y' + c'z',$$
$$\text{for } z, \ \ a''x' + b''y' + c''z' + \&\text{c.}$$

and substituting also for d^2x, d^2y, d^2z, δx, δy, δz, &c., their values in terms of these quantities, (a, b, c, a', &c. being supposed to be constant,) we obtain an equation of the same form as the preceding, for

$$a^2 + a'^2 + a''^2 = 1, \ \ ab + ac + bc = 0, \ \&\text{c. } \textit{see} \text{ page } 409,$$

the same substitutions being made in the expressions of the mutual distances, the coefficients a, b, c', a', &c., will disappear for the same reasons. The principle of D'Alembert by itself, without introducing the consideration of virtual velocities, would enable us to infer several important results; but it is its combination with that of virtual velocities which has contributed so much to the improvement of rational mechanics, as by means of it all mechanical problems are reducible to one sole principle, namely, that of virtual velocities; and thus every problem of dynamics may be reduced to the integrations of differential equations, so that as it belongs to pure analysis alone to effect the integration, the only obstacle to the perfect solution of every problem of dynamics arises from the imperfection of our analysis.

(p) In order to determine the condition of a fluid mass at each instant, we must know the direction of the motion of a molecule, its velocity, its pressure p, and the density ρ, but if we know the three partial velocities, parallel to the three ordinates, we shall have the entire ve-

locity, and also the direction, for the partial velocities divided by the entire velocities, express the cosines of the angles which the coordinates make with its direction; hence we have five unknown quantities. Now, in the general equation of equilibrium furnished in Notes, page 452, namely, $\delta p = P.\delta x + Q.\delta y + R.\delta z$, the characteristic δ is independent of the time; but when the fluid is in motion we must, by what has been just established, substitute

$$P - \frac{d^2x}{dt^2} \text{ for } P,\ Q - \frac{d^2y}{dt^2} \text{ for } Q,\ R - \frac{d^2z}{dt^2} \text{ for } R,$$

and after the substitution, if we concinnate, and assume that

$$P.\delta x + Q.\delta y + R.\delta z = \delta V,$$

then we shall have

$$\frac{\delta p}{\rho} = \delta V - \frac{d^2x}{dt^2}.\ \delta x - \frac{d^2y}{dt^2}.\ \delta y - \frac{d^2z}{dt^2}.\ \delta z;$$

since the variations δx, δy, δz, are independent, this equation is equivalent to three distinct equations; besides these, we obtain another from the circumstance of the continuity of the fluid, for though each indefinitely small portion of the fluid changes its form, and if it is compressible, its volume likewise, during the motion, still as the mass must be constant, the product of the volume into the density must be the same as at the commencement; and by equating the two values of the mass we obtain the equation relative to the continuity of the fluid.

Neglecting quantities indefinitely small of the fifth order, the volume of the element at the end of the time

$$t + dt \text{ is } dx.dy.dz.\left(1 + \frac{du}{dx}.\ dt + \frac{dv}{dy}.\ dt + \frac{dw}{dz}.\ dt\right),$$

and the density at the same epoch becomes

$$\rho + \frac{d\rho}{dt}.\ dt + \frac{d\rho}{dx}.\ udt + \frac{d\rho}{dy}.\ vdt + \frac{d\rho}{dz}.\ wdt,$$

multiplying this expression by the corresponding volume, the product expresses the mass at the end of $t + dt$, from which subtracting $\rho.dx.dy.dz$, the remainder will be the variation of this mass, which should be $= 0$; hence, we obtain by suppressing common factors, and neglecting dt^2

$$\frac{d\rho}{dt} + \frac{d\rho}{dx}.u + \frac{d\rho}{dy}.v + \frac{d\rho}{dz}w + \rho.\frac{du}{dz} + \rho.\frac{dv}{dy}$$
$$+ \rho.\frac{dw}{dz} = 0; \; i.\,e. \; \frac{d\rho}{dt} + \frac{d.\rho u}{dx} + \frac{d.\rho v}{dy} + \frac{d.\rho z}{dz} = 0;$$

if the fluid is incompressible this equation is resolvable into two, for both the mass and also the density remain the same. The two into which it is resolvable are

$$\frac{d\rho}{dt} + \frac{d\rho}{dx}.u + \frac{d\rho}{dy}.u + \frac{d\rho}{dz}w = 0,$$
$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0;$$

and these combined with the three equations already mentioned, will be sufficient to determine p, ρ, u, v, w, in a function of x, y, z, t; the first of these equations becomes a purely identical one when ρ is constant, but in this case we have only four unknown quantities. With respect to elastic fluids we have also only four different equations; however, it is to be remarked, that in this species of fluid, the density is always in a given ratio to the pressure p, therefore they are reduced to one unknown quantity, provided that the temperature is given; and even if it is not, if it varies according to a given law, so that that the temperature may be assumed a given function of x, y, z, and t, the coefficient which expresses the ratio of ρ to p, will be a given function of these variables; consequently, whether the motion to be determined be that of an incompressible fluid, or of one, of which the temperature is constant, or variable according to a given law, we shall in all cases have as many differential equations as unknown quantities; as these equations are those of partial differences between four independent variables,

x, y, z, and t, their perfect integrations cannot be effected by the ordinary methods, except that by means of some hypothesis they are simplified; and even in such a case, we should determine by means of the state of the fluid at the commencement of the motion, the arbitrary functions which their integrals contain.

(q) If in the expressions of p. 485, we suppose the origin of the coordinates to be in a point x, y, z, then in the values of $\delta f, \delta f', \delta f''$, &c., we have evidently $\delta x' = \delta x + \delta x_{,}'$, $\delta y = \delta y + \delta y_{,}'$, $\delta z' = \delta z + \delta z_{,}' +$ &c., therefore if in the values of δf, $\delta f'$, $\delta f''$, &c, of the variations of the mutual distances given in page 448, we substitute these values for $\delta x'$, $\delta y'$, $\delta z'$, &c., the variations δx, δy, δz, &c. will disappear from these expressions; consequently, by substituting these values for $\delta x'$, $\delta y'$, $\delta z'$, &c. in the equation given in page 485, we obtain

$$0 = m.\delta x.\left(\frac{d^2x}{dt^2} - P\right) + m.\delta y\left(\frac{d^2y}{dt^2} - Q\right) + m.\delta z\left(\frac{d^2z}{dt^2} - R\right)$$

$$+ m'\delta x.\left(\frac{d^2x'}{dt^2} - P'\right) + m'\delta x_{,}'.\left(\frac{d^2x'}{dt^2} - P'\right) + m'.\delta y.$$

$$\left(\frac{d^2y'}{dt^{2\prime}} - Q'.\right) + m'\delta y_{,}'\left(\frac{d^2y'}{dt^2} - Q'.\right) + \text{\&c.}$$

the terms in the expression which are multiplied by δx, δy, δz, respectively, are, by adding them together

$$\Sigma m.\left(\frac{d^2x}{dt^2} - P\right); \quad \Sigma m.\left(\frac{d^2y}{dt^2} - Q\right); \quad \Sigma m.\left(\frac{d^2z}{dt^2} - R\right)$$

Consequently, if, as is supposed, the system be free, the conditions relative to the mutual connexion of the bodies will only depend on their mutual distances, hence the variations of δx, δy, δz, are independent of these conditions, and therefore the preceding expressions by which they are respectively multiplied, must be put severally equal to cypher; and as from what is laid down in page 446,

$$A = \frac{\Sigma mx}{\Sigma m}; \; B = \frac{\Sigma my}{\Sigma m}; \; C = \frac{\Sigma mz}{\Sigma m},$$

we have

$$\frac{d^2A}{dt^2}=\frac{\Sigma m.\frac{d^2x}{dt^2}}{\Sigma m}=\frac{\Sigma m.P}{\Sigma m}.$$

we obtain in the same manner

$$\frac{d^2B}{dt^2}=\frac{\Sigma m.Q}{\Sigma m};\quad \frac{d^2C}{dt^2}=\frac{\Sigma m.R}{\Sigma m};$$

therefore if all the bodies of the system were united in the centre of gravity, and the forces which are applied to them, when separate, were simultaneously impressed on them; the motion of such a body is the same as that of the centre of gravity; if the system was only subject to the mutual actions p, p', &c., of the bodies composing it, and to their reciprocal attractions; then since f, f', f'', &c. the distances of the bodies are

$$=\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2},$$
$$\sqrt{(x''-x)^2+(y''-y)^2+(z''-z)^2}+\text{\&c.}$$

in consequence of the sole action p, we have

$$mP=p.\frac{(x-x')}{f},\quad mQ=p.\frac{(y-y')}{f},\quad mR=p.\frac{(z-z')}{f};$$
$$m'P'=p.\frac{(x'-x)}{f},\quad m'Q'=p.\frac{(y'-y)}{f},\quad m'R'=p.\frac{(z'-z)}{f}.$$
$$\therefore mP+m'P'=0;\quad mQ+m'Q'=0;\quad mR+m'R'=0;\ \text{\&c.}$$

and a similar proof may be shewn for the bodies in the case of their mutual attractions. As action is equal to reaction, though its direction be contrary, when two bodies impinging on each other exercise a *finite* action in an instant, their reciprocal action will disappear in the expressions $\Sigma.mP$, $\Sigma.mQ$, &c.; in fact, as we can always suppose the action of the bodies to be effected by means of a spring interposed between them, which endeavours to restore itself after the shock, the effect of the shock will be produced by forces of the same nature with p, which, as we have seen, disappear in the expressions $\Sigma.mP$, $\Sigma.mQ$,

&c. By integrating $\frac{d^2A}{dt^2}=0$, we obtain $\frac{dA}{dt}=b$, and $A=bt+a$; a is the value of A at the commencement of the motion, and b is the uniform velocity of the centre of gravity resolved parallel to A. In like manner the invariability of the motion of the centre of gravity of a system of bodies, notwithstanding their mutual action, subsists even in the case in which some of the bodies lose in an instant, by this action a finite quantity of motion; for since $d.\frac{A}{dt}.\ \Sigma m=\Sigma m.\frac{dx}{dt}=$ the quantity of motion, and since by the principle of D'Alembert the quantity of motion $\Sigma m.\frac{dx}{dt}$ before and after impact, should be equal to cypher, *i. e.* such as would cause an equilibrium in the system, it follows that $\frac{dA}{dt}.\ \Sigma m$, before and after impact should be equal to nothing, *i. e.* as Σm is given, $\frac{dA}{dt}$, the velocity of the centre of gravity in the direction of the axis of x, is not affected by the impact. We can therefore always determine the motion and direction of the centre of gravity of a system, by the law of the composition of forces, for it moves in the same manner as a body equal to the sum of the bodies would move, provided that the same momenta are communicated to it as are impressed on the respective bodies of the system; and if the several bodies of the system were only subject to their mutual action, then they would meet in the centre of gravity, for they must meet, and the centre of gravity remains at rest.

(*t*) We may make the variation δx disappear from the expressions for δf, $\delta f'$, &c., by another supposition beside that of page 489; for if we assume

$$\delta x'=y'\frac{\delta x}{y}+\delta x_{,}'\,;\ dx''=\frac{y''\delta x}{y}+\delta x_{,}''\,;\ \delta y=-\frac{x\delta x}{y}.$$
$$+\delta y_{,},\ \&c.$$

then if in the expression for δf, $\delta f'$, &c, $=$

$$\frac{\sqrt{(x'-x).(\delta x'-\delta x)+(y'-y).(\delta y'-\delta y)+(z'-z(.(\delta z'-\delta z)}}{f},$$

&c. we substitute for $\delta x'$, $\delta x''$, $\delta y,'$ $\delta y''$, &c. their preceding values, they become

$$(x'-x).\left(\frac{y'\delta x}{y}+\delta x_{,}'-\delta x\right)+(y'-y)\left(-\frac{x'\delta x}{y}+\delta y_{,}'+\frac{x\delta x}{y}-\delta y_{,}\right)\text{ divided by } f, =$$

by omitting quantities which destroy each other

$$(x'-x),\ \delta x_{,}'+(y'-y).\ (\delta y_{,}'-\delta y_{,}),\ \&c.$$

hençe, making those substitutions, the variation δx disappears from the expressions for δf, $\delta f'$, &c., and it is easy to perceive, if the preceding values be substituted for δx, δy, $\delta x''$, &c., in the equation given in page 485, that the coefficient of δx will be

$$\Sigma.m.\frac{(xd^2y-yd^2x)}{dt^2}+\Sigma m.\ (\mathrm{P}y-\mathrm{Q}x),$$

which is, by what precedes, equal to cypher, therefore its integral with respect to t is

$$c=\Sigma m.\ \frac{(xdy-ydx)}{dt}+\Sigma\int.m(\mathrm{P}y-\mathrm{Q}x).\ dt,$$

if in place of the forces Q, Q′, &c., parallel to the axis of y, we substitute the forces R, R′, &c., parallel to the axis of z, or in this last if we substitute Q, Q′, for P, P′, &c., we shall obtain the corresponding equations

$$c'=\Sigma m.\ \frac{(xdz-zdx)}{dt}+\Sigma\int m.(\mathrm{P}z-\mathrm{R}x).dt,$$

$$c''=\Sigma m.\ \frac{(ydz-zdy)}{dt}+\Sigma\int m.(\mathrm{Q}z-\mathrm{R}y).\ dt;$$

it is evident, from what precedes, if the bodies of the system are only subjected to the action of forces arising

from their mutual action, and of forces directed to a fixed point, that then

$\Sigma m.(\mathrm{P}y-\mathrm{Q}x)$, $\Sigma m.(\mathrm{P}z-\mathrm{R}x)$ + &c. are respectively $=0$;

$$\because c=\Sigma m.\ \frac{xdy-ydx)}{dt};\ c'=\Sigma m.\ \left(\frac{xdz-zdx}{dt}\right)\text{\&c. but}$$

from what has been stated in pages 390, 429, $\frac{xdy-ydx}{2}$, = the area traced by the radius vector of m in dt, hence then appears the truth of what is asserted in the text, that when the bodies composing the system are only subject to their mutual actions, and to attractions directed towards a fixed point, the sum of the areas multiplied respectively by the masses of the bodies is proportional to the time. The constant quantities c, c', c'', may be determined at any instant, when the velocities and coordinates of the bodies are given at that instant. There are three cases in which this principle of the conservation of areas obtains, when the forces are only the result of the mutual action of the bodies composing the system, when the forces pass through the origin of the coordinates, when the system is moved by an initial impulse ; in the first and last cases the origin of the coordinates may be any point whatever; if there is a fixed point in the system, as by what is stated in page 412, the principle of the conservation of areas may be reduced to that of moments, the principle obtains when this point is made the origin of the coordinates; for in that case, $\mathrm{P}y-\mathrm{Q}x$, which is the moment with respect to the origin, will disappear, *see* Notes, page 412; if there are two *fixed* points in the system, only one of the three equations obtains, namely, that which contains those coordinates, the plane of which is perpendicular to the line joining the given points. If all the bodies of the system are equal, the theorem comes to this, that the sum of the areas traced by the radii vectores about the focus is proportional to the times.

(*t*) If the variations δx, δy, dz, $\delta x'$, &c., be supposed equal to dx, dy, dz, dx', &c., which supposition we are permitted to make, the equation given in page 481, becomes

$$0 = mdx.\left(\frac{d^2x}{dt^2} - \text{P.}\right) + mdy.\left(\frac{d^2y}{dt^2} - \text{Q.}\right) + m.dz.\left(\frac{d^2z}{dt^2} - \text{R}\right)$$

$$+ m'.dx'.\left(\frac{dx'^2}{dt^2} - \text{P.}\right) + m'dy'.\left(\frac{d^2y'}{dt^2} - \text{Q}'\right) + m'\,dz'.\left(\frac{d^2z}{dt^2} - \text{R}'.\right)$$

+ &c., of which the integral is

$$\Sigma m.\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = c_{\prime}' + \Sigma.fm.(\text{P}dz + \text{Q}dy + \text{R}dz),$$

this last term is an exact integral, if the forces P, Q, R, P′, &c. are the results of attractive forces directed towards fixed centres and of a mutual attraction between the bodies, which is some function of the distance; if we suppose it = to ϕ, the preceding equation will become $\Sigma mv^2 = c + 2\phi$, see page 432; hence, if the bodies composing the system are not solicited by any forces, ϕ vanishes, and $\Sigma mv^2 = c$, *i. e.* the sum of the living forces is constant, and if it does not vanish, the sum of the increments of the living forces is the same, whatever be the nature of the curves described, provided that their points of departure and arrival are the same. What has been stated respecting the mutual attraction of the bodies of the system, is equally true respecting repulsive forces, which vary as some function of the distance; it is also true, when the repulsions are produced by the action of springs interposed between the bodies, for the force of the spring must vary as some function of the distance between the points; hence in the impact of perfectly elastic bodies, though the quantity of motion communicated may be *increased indefinitely*, still the *vis viva* after the impact remains the same as before; indeed, *during* the impact, the *vis viva* varies as the coordinates of the respective points vary, but after the restitution of the bodies, from their perfect elasticity they resume their original position, and therefore the value of the *vis viva* remains the same as before: but if the elasticity be not perfect, in

order to have the *vis viva* at any instant, we should have the relation which exists between the compressive and restitutive force. The *vis viva* of a system is evidently diminished when the motion is modified by friction, or the resistance of a medium, for in that case $(\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz)$ is not a perfect integral.

It is evident from the manner in which the principle of the *vis viva* was deduced, that it only obtains when the motions of the bodies change by imperceptible gradations, if these motions undergo abrupt changes, the living force is diminished by a quantity which is thus determined, let $\Delta.\frac{dx}{dt}$. $\Delta.\frac{dy}{dt}$, &c. denote the differences of $\frac{dx}{dt}, \frac{dy}{dt}$, &c, from one instant to another, and from the principle of D'Alembert, as $\Delta.\frac{dx}{dt}$ is the variation of the velocity on the supposition that the body is entirely free, and $\mathrm{P}.dt$, the variation which actually takes place, in consequence of the actions of the bodies of the system, we may apply the reasoning of page 483 to this case; therefore the following equations obtain

$$\Sigma m.\left(\Delta.\frac{dx}{dt}.\frac{\delta x}{dt} + \Delta.\frac{dy}{dt}.\frac{\delta y}{dt} + \Delta.\frac{dz}{dt}.\frac{\delta z}{dt} + \&c.\right)$$
$$- \Sigma m.(\mathrm{P}\delta x + \mathrm{Q}\delta y + \mathrm{R}\delta z) = 0;$$

now as dx, dy, dz become in the subsequent instants

$$dx + \Delta.dx,\ dy + \Delta.dy,\ dz + \Delta.dz,\ \&c.$$

if we assume $\delta x, \delta y, \delta z$, &c.= to these quantities, we evidently satisfy the condition of the connexion of the parts of the system, therefore substituting these quantities for δx, δy, δz, &c., the preceding equation becomes

$$\Sigma m.\left\{\left(\frac{dx}{dt} + \Delta.\frac{dx}{dt}.\right)\Delta.\frac{dx}{dt} + \left(\frac{dy}{dt} + \Delta.\frac{dy}{dt}\right)\Delta.\frac{dy}{dt} +\right.$$

$$\left(\frac{dz}{dt}+\Delta.\frac{dz}{dt}\right)\Delta.\frac{dz}{dt}\Big\} + \&c.$$

$$-\Sigma m.P(dx+\Delta\, dx)+Q.(dy+\Delta.dy)+R.(dz+\Delta.dz)\&c.,$$

the integral of $mP.(dx+\ \Delta.dx)$ is evidently equal to $\int m.$ $P.dx$, &c., and the integral of

$$m.\frac{dx}{dt}.\ \Delta.\frac{dx}{dt}=m.\frac{dx^2}{dt^2},$$

for $\Delta.(x^2) = 2xh + h^2$, and if h be made equal to Δx, it becomes

$2x\Delta.x+(\Delta x^2)$, $\because$ $2S.(x\Delta x+(\Delta x)^2)=S.(2x\Delta x+(\Delta x)^2$

$+S.(\Delta x)^2=x^2+S.(\Delta x)^2$; *see* Lacroix, tom. 3, No. 344, therefore if we multiply the preceding equation by 2, and substitute dx, in place of x, we shall obtain after concinnating

$$\Sigma m.\left(\frac{dx^2+dy^2+dz^2}{dt^2}\right)-2\int m.(Pdx+Qdy+Rdz)$$

$$+S.\Sigma m.\left\{\left(\Delta.\frac{dx^2}{dt^2}\right)+\left(\Delta.\frac{dy^2}{dt^2}\right)+\left(\Delta.\frac{dz^2}{dt^2}\right)\right\};$$

i. e. if v, v', v'', &c., denote the velocities of the several bodies m, m', m'', &c., we have

$$\Sigma.mv^2=C.+2\Sigma\int m.(Pdx+Qdy+Rdz)$$

$$-S.\Sigma m.\left\{\left(\Delta.\frac{dx}{dt}\right)^2+\Delta.\left(\frac{dy}{dt}\right)^2+\left(\Delta.\frac{dz}{dt}\right)^2\right\},$$

as the quantity under the sign S is always positive, the living force of the system is diminished by the mutual action of the bodies as often as $\Delta.\frac{dx}{dt}$ is finite, as

$\frac{dx^2+dy^2+dz^2}{dt^2}$ expresses the square of the velocity of

m before the shock, and

$$\frac{(dx+\Delta.dx)^2+(dy+\Delta.dy)^2+(dz+\Delta dz)^2}{dt^2},$$

the square of the velocity of m after the shock; and since from the principle of D'Alembert,

$$\Sigma m.(2dx\,\Delta.dx + 2(\Delta dx)^2 + 2dy.\ \Delta.dy + 2(\Delta.dy)^2 + 2dz.\Delta.dz + 2(\Delta.dz)^2) = 0,$$

If we subtract this from the preceding expression the difference becomes

$$= \Sigma m.\frac{dx^2 + dy^2 + dz^2}{dt^2} - \Sigma m.\frac{(\Delta.dx)^2 + (\Delta.dy)^2 + (\Delta.dz)^2}{dt^2},$$

$$\text{and as } \Sigma m.\ \frac{(dx^2 + dy^2 + dz^2)}{dt^2} = \Sigma m v^2,$$

the living force of the system before the shock,

$$\frac{(\Delta.dx)^2 + (\Delta.dy)^2 + (\Delta.dz)^2}{dt^2} = V^2 =$$

the square of the velocity lost by the shock, and $\Sigma.m V^2$(= the loss which the *vis viva* sustains by the shock) is equal to the sum of the living forces which would belong to the system, if each body was solely actuated by that which is lost by the shock. This theorem was first announced by Carnot.

(s) The variation of the *vis viva* of the system is equal to

$$2\Sigma m.(P.dx + Q.dy + R dz) = d.(\Sigma m.v^2),$$

therefore when this expression vanishes, $\Sigma m v^2$ is either a maximum or a minimum; but from the principle of virtual velocities it appears that when P, Q, R, P′, &c., constitute an equilibrium

$$P.\delta x + Q.\delta y + R.\delta z + P'.\delta x' + \&c. = 0;$$

and when δx, δy, δz, &c., are subjected to the conditions of the connexion of the parts of the system, we may substitute dx, dy, dz for these variations; consequently, we have

$$\Sigma m.(P.dx + Q.dy + R.dz),$$

the variation of the *vis viva* equal to nothing in this

case, and therefore the *vis viva* is either a maximum or minimum. If the system was slightly disturbed from the position of equilibrium, expressing P, Q, R, &c., in terms of the coordinates and expanding the resulting expressions into a series proceeding according to the variations of the coordinates; the first term of the series will be the value of ϕ when the system is in equilibrio; and since it is given, it may be made to coalesce with $c_{/}'$ in the expression given in page 494; the second term vanishes by the conditions of the problem; and when Σmv^2 is a maximum, the theory of maxima and minima shews that the third term of the expansion may be made to assume the form of a sum of squares affected with a negative sign, *see* Lacroix, No. 134, the number of terms in this sum being equal to the number of variations or independent variables.

The terms whose squares we have assumed, are linear functions of the variations of the coordinates, and vanish at the same time with them; and they are greater than the sum of all the remaining terms of the expansion. The constant quantity being equal to $c_{/}' +$ the value of Σmv^2, when P, Q, R, P′, &c., constitute an equilibrium, it is necessarily positive, and may be rendered as small as we please by diminishing the velocities; but it always exceeds the greatest of the quantities whose squares have been substituted in place of the variations of the coordinates; for if it were less, this negative quantity would exceed the constant quantity, and therefore render the value of Σmv^2, negative; consequently, these squares and the variations of the coordinates, of which they are linear functions, always remain very small, therefore the system will always oscillate about the position of equilibrium, and hence this equilibrium will be one of stability. But in the case of ϕ being a minimum, it is not requisite that the variations should be always constrained to be very small in order to satisfy the equation of living forces; this indeed does not prove that

there is no limit then to those variations, which should be done in order to prove the equilibrium to be instable; in order to demonstrate this, we should substitute for these variations their values in a function of the time, and then shew from the form of those functions, that they increase indefinitely with the time, however small the primitive velocities may be.

Let P, P′, P″, &c., denote the weights of any number of bodies in equilibrio, and z, z', z'', &c., their coordinates with respect to an horizontal plane; then if the position of the system be disturbed by any quantity, however small, we have

$$\mathrm{R}.\delta z+\mathrm{R}'.\delta z'+\mathrm{R}''.\delta z''+\&c.=0;\ \therefore \frac{\mathrm{R}z+\mathrm{R}z'+\mathrm{R}''z''}{\Sigma.\mathrm{R}}+\&c.$$

(which is equal to $z_{,}$ the distance of the centre of gravity of all the bodies of the system from the horizontal plane) is either a maximum or a minimum; and the sum of the living forces is a maximum when the centre ceases to descend, and commences to ascend, for

$$\Sigma m.(\mathrm{P}.dx+\mathrm{Q}.dy+\mathrm{R}.dz),$$

in this case becomes $\Sigma m.\mathrm{R}.dz$; and therefore by substitution we have $\Sigma m.v^2=c_{,}'+z_{,}\mathrm{R}.\Sigma m$, consequently Σmv^2 is a maximum or minimum, according as $z_{,}$ is a maximum or minimum; when Σmv^2 is a maximum the equilibrium is stable, when a minimum the equilibrium is instable. For from the definition of stability, it appears that then the bodies tend to revert to the position of equilibrium, therefore the velocities will diminish according as the system deviates more from the position of equilibrium, consequently the sign of the second differential of ϕ will be negative; hence Σmv^2 will be a maximum in this case, and in the contrary it will be evidently a minimum.

Let, as in page 413, F the force be $\div$1 to $\phi(v)$, then this force resolved parallel to the axes of x, y, z, becomes respectively

$$= \phi.(v).\ \frac{dx}{ds},\ \phi.(v).\left(\frac{dy}{ds}\right),\ \phi.(v).\ \left(\frac{dz}{ds}\right);$$

moreover, the forces at the subsequent instant are

$$\phi.(v).\ \frac{dx}{ds} + d.\left(\phi.(v).\ \frac{dx}{ds}\right),\ \phi.(v).\ \frac{dy}{ds}\ ,$$

$$+ d.\left(\phi.(v).\frac{dy}{ds}\right),\ \ \phi.(v).\frac{dz}{ds} + d.\ \left(\phi.(v).\ \frac{dz}{ds}\right) + \text{\&c.}$$

if P, Q, R, P′, &c., denote the same quantities as before, the system will, by what is established in page 485, be in equilibrio in consequence of these forces and the differentials

$$d.\left(\frac{dx}{dt}.\ \ \frac{\phi.(v)}{v}\right),\ d.\left(\frac{dy}{dt}.\ \ \frac{\phi.(v)}{v}\right),\ \ d.\ \left(\frac{dz}{dt}.\ \ \frac{\phi.(v)}{v}\right),$$

taken with a contrary sign, therefore in place of the equation given in page 485, we shall have the following

$$0 = \Sigma m.\ \left(\delta x.d.\left(\frac{dx}{dt}\ .\ \ \frac{\phi.(v)}{v}\right) - \text{P}.dt.\right)$$

$$+ \delta y d.\ \left(\frac{dy}{dt}.\ \frac{\phi.(v)}{v}\right) - \text{Q}.dt.\Big) + \delta z.d.\left(\frac{dz}{dt}.\frac{\phi.(v)}{v}\right) - \text{R}.dt,\Big),$$ it

differs from that equation in this respect, that $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, are multiplied by the function $\frac{\phi\ (v)}{v}$, which in the case of the force ÷l to the velocity is = to unity; this difference renders the solution of problems extremely difficult; however we may obtain from the preceding equation principles analogous to those of the conservation of living force, of areas, and of the motion of the centre of gravity. For instance, the preceding expression, by changing δx, δy, δz, &c., into dx, dy, dz, &c., becomes

$$\Sigma m.\ \left(\ dx.d.\ \left(\frac{dx}{ds}.\ \phi.\ (v).\right) + dy.d.\left(\frac{dy}{ds}.\ \ \phi.(v).\right)\right.$$

$$\left. + dz.d.\ \left(\frac{dz}{ds}.\ \phi.)v)\ \right)\right)$$

i. e. by expanding the expression

$$= \Sigma m. \frac{(dx.d^2x + dy.d^2y + dz.d^2z)}{ds}. \phi.(v)$$

$$- \Sigma m. \frac{(dx^2 + dy^2 + dz^2)}{ds^2}. d^2s. \phi.(v) +$$

$$\Sigma m. \left(\frac{dx^2 + dy^2 + dz}{ds}. d.\phi.(v) = \Sigma m.d^2s.\phi.(v) - \Sigma m.d^2s.\phi.(v)\right.$$

$$+ \Sigma m.ds.d.\phi(v).$$

and this last quantity is equal by substitution to

$$\Sigma m.vdt.dv.\phi'.(v),$$

therefore we have

$$\Sigma f mv.dv.\phi'.(v) = c_{\prime}' + \Sigma f m.(P.dx + Q.dy + R.dz);$$

if this last term is an exact differential equal to $d\lambda$, we shall have

$$\Sigma.f mvdv.\phi'.(v) = c_{\prime}' + \lambda;$$

an equation which establishes what is stated in page 292. If as in page 489, we make

$$\delta x = \delta x + \delta x_{\prime}', \quad \delta y = \delta y + \delta y_{\prime}', \quad \delta z = \delta z + \delta z_{\prime}',$$

and make, as in that page, the coefficients of δx, δy, δz, respectively equal to cypher, we shall have

$$0 = \Sigma m. \left(d. \left(\frac{dx}{dt}. \frac{\phi.(v)}{v}\right) - P.dt.\right)$$

$$0 = \Sigma m. \left(d. \left(\frac{dy}{dt}. \frac{\phi.(v)}{v}\right) - Q.dt.\right);$$

$$0 = \Sigma m. \left(d. \left(\frac{dz}{dt}. \frac{\phi.(v)}{v}\right) - R.dt.\right),$$

which are analogous to those of page 464, from which the conservation of the motion of the centre of gravity was inferred, when the system is only subject to the mutual action and reciprocal attraction of the bodies composing it, in which case $\Sigma m P$, $\Sigma m Q$, $\Sigma m R$ are respectively equal to cypher; we can infer from the preceding equation

$$C = \Sigma m. \frac{dx}{dt}. \frac{\phi.(v)}{v}; \quad C' = \Sigma m. \frac{dy}{dt}. \frac{\phi.(v)}{v};$$

$C'' = \Sigma.m.\frac{dz}{dt}.\frac{\phi.(v)}{v}$; but $m.\frac{dz}{dt}.\frac{\phi.(v)}{v} = m.\phi.(v).\frac{dz}{ds} =$ the finite force of the body resolved parallel to the axis of z; provided that we understand by the force of a body, the product of the mass into that function of the velocity which expresses it; consequently in the preceding case the sum of the finite forces of the bodies composing the system is constant, whatever may be the nature of ϕ; but unless $\frac{\phi.(v)}{v} = 1$, the motion of the centre of gravity will not be uniform and rectilinear, for it is only in that case that we could prove from the expression $C = \Sigma m.\frac{dx}{dt}.\frac{\phi.(v)}{v}$, that dA, the differential of the coordinate of the centre of gravity, was constant.

Making the substitutions indicated in page 491, and afterwards putting the coefficient of $\delta x = 0$, we obtain, when the system is not actuated by extraneous forces,

$$0 = \Sigma m.\left(x.d.\left(\frac{dy}{dt}.\frac{\phi.(v)}{v}\right) - y.d.\left(\frac{dx}{dt}.\frac{\phi.(v)}{v}\right)\right) + \Sigma m.(Py - Qx).dt,$$

and by integrating

$$c = \Sigma m.\left(\frac{x\,dy - ydx}{dt}\right).\frac{\phi.(v)}{v} + \Sigma.\textstyle\int m.(Py - Qx).dt,$$

and in like manner

$$c' = \Sigma m.\left(\frac{x.dz - zdx}{dt}\right).\frac{\phi.(v)}{v} + \Sigma\textstyle\int m.(Pz - Rx).dt;$$

$$c'' = \Sigma m.\frac{ydz - zdy}{dt}.\frac{\phi.(v)}{v} + \Sigma\textstyle\int m.(Qz - Ry).dt,$$

and since, by what has been stated above, $m.\left(x.\frac{dy}{dt} - y.\frac{dx}{dt}\right).\frac{\phi.(v)}{v}$ is the moment of the finite force by which the body

is actuated, $\Sigma m\left(\frac{xdy-ydx}{dt}\right).\ \frac{\phi.(v)}{v}$ expresses the sum of the moments of all the finite forces of the bodies of the system to make it revolve about the axis of z, which, when $Py - Qx = 0$, is constant, and it evidently vanishes in the case of equilibrium.

(t) If the equation

$$\Sigma m.v^2 = C + 2\Sigma m.(P.dx + Q.dy + R.dz)$$

be differentiated with respect to the characteristic δ we shall have

$$\Sigma m.v\delta v = \Sigma m.(P.\delta x + Q.\delta y + R.\delta z),$$

and the equation given in page 485 then becomes

$$0 = \Sigma m.\left(\delta x.d.\frac{dx}{dt} + \delta y.d.\frac{dy}{dt} + \delta z.d.\frac{dz}{dt}\right)$$

$$-\Sigma.m.dt.vdv;$$ and as $vdt = ds$, $v'dt = ds'$, &c. we can obtain by the same process as in page 439,

$$\Sigma m.\delta.(vds) = \Sigma m.\ d.\left(\frac{dx.\delta x + dy.\delta y + dz.\delta z}{dt}\right),$$

integrating with respect to d, and extending the integrals to the entire curves described by the bodies m, m', &c. we shall have

$$\Sigma.\delta.\int m.vds = C + \Sigma m.\left(\frac{dx.\delta x + dy.\delta y + dz.\delta z}{dt}\right),$$

C, and also the variations δx. δy, δz, $\delta x'$, &c. refer to the extreme points of the curves described, and when these are invariable, we have $0 = \Sigma.\delta.\int mv.ds$, therefore $\Sigma.\int m.vds$ is a *minimum*. This expression becomes, by substituting for ds, ds', &c. $v.dt$, $v'.dt$, &c. $= \Sigma\int mv^2.dt$ = the sum of the living forces of the bodies composing the system, consequently, the principal of the least action, in fact, indicates that the sum of the living forces of the bodies composing the system is, in its transit from one position to another, a minimum: and when the bodies are not actuated by any accelerating forces, the velocities v, v', &c.

and the sum of the living forces are constant at each instant, see page 439;

$$\because \Sigma \int mv^2.dt = \Sigma mv^2.\int dt,$$

and the sum of the living forces for any interval of time is $\div$l to this time, consequently in this case the body passes from one position to another in the shortest possible time. As $\Sigma\int m.vds = \Sigma\int m.v^2.dt$, La Grange proposed to alter the denomination of the principle of least action, and to term it the principle of the greatest or least living force. The advantage from this mode of expression would be, that it is equally applicable to a state of equilibrium and motion, since, in the state of equilibrium, it has been already shewn to be either a maximum or minimum.

(*w*) This is evident from what goes before, for from the principle of action and reaction the expressions

$$\Sigma.mP, \text{ \&c. } \Sigma m.(Py - Qx)\text{\&c.} = 0,$$

whatever changes are produced by the mutual actions of the bodies. Let X, Y, Z represent the coordinates of the moveable origin of the coordinates,

$$x = X + x_{,}; \; y = Y + y_{,}; \; z = Z + z_{,}; \; x' = X + x'_{,}; \text{ \&c.}$$

If the origin moves with a uniform rectilinear motion

$$d^2X, = 0, \; d^2Y = 0, \text{ \&c.}$$

therefore substituting for d^2x, we have, when the system is free, by the nature of the centre of gravity,

$$\Sigma m.(d^2X + d^2x_{,}) - \Sigma m.P.dt^2 = 0,$$
$$\Sigma m.(d^2Q + d^2y) - \Sigma m.Q.dt^2) = 0, \text{ \&c.}$$

by substituting

$$\delta X + \delta x_{,}, \; \delta Y + \delta y_{,}, \text{ \&c.}$$

in place of δx, δy, &c. in the equation of page 485, we shall have

$$0 = \Sigma m.\delta x_{,}.\left(\frac{d^2x_{,}}{dt^2} - P.\right) + \Sigma m.\delta y_{,}.\left(\frac{d^2y_{,}}{dt^2} - Q.\right)$$
$$+ \Sigma m.\delta z_{,}.\left(\frac{d^2z_{,}}{dt^2} - R.\right),$$

which is precisely of the same form as the equations given in page 485, and the same consequences may evidently be derived from them; if X, Y, Z denote the coordinates of the centre of gravity, by the nature of it we have

$$\Sigma m x_{,} = 0, \ \Sigma m y_{,} = 0, \ \Sigma m z_{,} = 0,$$

$$\because \ \Sigma m. \left(\frac{xdy - ydx}{dt}\right) = \frac{\mathrm{X}dy - \mathrm{Y}dx}{dt}. \ \Sigma m +$$

$$\Sigma m. \ \frac{(x_{,}.dy_{,} - y_{,}.dx_{,})}{dt},$$

in like manner

$$\Sigma m. \left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = \frac{d\mathrm{X}^2 + d\mathrm{Y}^2 + d\mathrm{Z}^2}{dt^2}. \ \Sigma m +$$

$$\Sigma m. \left(\frac{dx_{,}^2 + dy_{,}^2 + dz_{,}^2}{dt^2}\right),$$

for $\Sigma m.dx^2 = \Sigma m.d\mathrm{X}^2 + 2\Sigma m.dx_{,}.\ d\mathrm{X} + \Sigma m.dx_{,}^2$,

and as $2d\mathrm{X}.\Sigma m.dx_{,} = 0$, we have

$$\Sigma m.dx^2 = d\mathrm{X}^2.\Sigma m + \Sigma m.\ dx_{,}^2.$$

therefore it appears, that if the origin be transferred from another point to the centre of gravity, the quantities which result are composed of two different expressions, namely of those which would obtain if all the bodies of the system were concentrated in the centre of gravity; and secondly, of quantities relative to the centre of gravity supposed fixed; and since the first described quantities are constant, the reason why the principles in question obtain, with respect to the centre of gravity is evident; also if the origin of the coordinates be supposed in this point, the plane which passes through it, and relatively to which $\Sigma m. \left(\frac{xdy - ydx}{dt}\right)$ is a maximum, remains always parallel to itself during the motion of the system, and the same function relatively to every other plane perpendicular to it, vanishes, see note (x), and page 509.

(x) There exists a plane with respect to which c' and c'', in page 492 vanish, which is thus determined, let θ represent the inclination of the required plane formed by two of the new axes x'', y'' with the plane of x, y, and let ψ represent the angle between the axis of x and the intersection of x'', y'' with x, y, and ϕ the angle between x'', and the intersection of x, y, x'', y'', then by substituting it would be easy to shew that

$$x'' = x.(\cos.\theta.\sin.\psi.\sin.\phi + \cos.\psi.\cos.\phi) +$$
$$y.(\cos.\theta.\cos.\psi.\sin.\phi - \sin.\psi.\cos.\phi) - z.\sin.\theta.\sin.\phi;$$
$$y'' = x.(\cos.\theta.\sin.\psi.\cos.\phi - \cos.\psi.\sin.\phi) +$$
$$y.(\cos.\theta.\cos.\psi.\cos.\phi + \sin.\psi.\sin.\phi) - z\sin.\theta.\cos.\phi.$$
$$z'' = x.\sin.\theta.\sin.\psi + y.\sin.\theta.\cos.\psi + z.\cos.\theta;$$

if we take the expressions $x''dy'' - y''dx''$, by substituting for $y''dx''$, $x''dy''$, &c. their values, neglecting quantities which destroy each other, and observing that $xdy - ydx = c$, $xdz - zdx = c'$, &c. we shall obtain after all substitutions

$$\Sigma m.\left(\frac{x''.dy'' - y''.dx''}{dt}\right) = c.\cos.\theta - c'.\sin.\theta.\cos.\psi + c''.$$
$$\sin.\theta.\sin.\psi;\ \Sigma m.\left(\frac{x''.dz'' - z''.dx''}{dt}\right) = c.\sin.\theta.\cos.\phi.$$
$$+ c'.(\sin.\psi\sin.\phi + \cos.\theta.\cos.\psi.\cos.\phi) + c''.(\cos.\psi.\sin.$$
$$\phi - \cos.\theta.\sin\psi.\cos.\phi),\ \Sigma m.\left(\frac{y''dz'' - z''dy''}{dt}\right) = -c.\sin.$$
$$\theta.\sin.\phi + c'.(\sin.\psi.\cos.\phi - \cos.\theta.\cos.\psi.\sin.\phi) + c''.\cos.$$
$$\psi.\cos.\phi + \cos.\theta.\sin.\psi.\sin.\phi),$$

if θ and ψ are so determined that $\sin.\theta.\sin.\psi =$

$$\frac{c''}{\sqrt{c^2 + c'^2 + c''^2}};\ \sin.\theta.\cos.\psi = \frac{-c'}{\sqrt{c^2 + c'^2 + c''^2}};$$

and therefore $\cos.\theta = \dfrac{c}{\sqrt{c^2 + c'^2 + c''^2}}$, we shall have

$$\Sigma m.\left(\frac{x''.dy'' - y''.dx''}{dt}\right) = \sqrt{c^2 + c'^2 + c''^2}$$

$$\Sigma m.\left(\frac{x''.dz''-z''.dx''}{dt}\right)=0;\quad \Sigma m.\left(\frac{y''.dz''-z''.dy''}{dt.}\right)=0;$$

therefore with respect to a plane determined in this manner, c', c'' vanish; there exists only one plane which possesses this property, for supposing it to be the plane of x, y, then

$$\Sigma m.\left(\frac{x''.dz''-z''.dx''}{dt}\right)=c.\sin.\theta.\cos.\phi;$$

$$\Sigma m.\left(\frac{y''.dz'-z''.dy''}{dt}\right)=-c.\sin.\theta.\sin.\phi;$$

if these two functions be put = to cypher, we shall have $\sin.\theta=0$; therefore the plane x'', y'', coincides with the plane x, y; since whatever has been the direction of the original plane x, y, the value of

$$\Sigma m.\left(\frac{x''.dy''-y''.dx''}{dt}\right) \text{ is } \sqrt{c^2+c'^2+c''^2},$$

it follows that $c^2+c'^2+c''^2$ is constant, and that the plane of x'', y'', determined by what precedes, is that with respect to which $\Sigma m.\left(\frac{x''.dy''-y''.dx''}{dt}\right)$ is a maximum: this plane therefore possesses these remarkable properties, namely, that the sum of the areas traced by the projections of the radii vectores of the several bodies on it, and multiplied by their masses, is the greatest possible, and that the same sum vanishes for every plane which is perpendicular to it; by means of these properties we can always find its position, whatever variations may be induced in the respective positions of the bodies in consequence of their mutual action; as $\cos\theta$, $\sin.\theta.\cos.\psi$, $\sin.\theta.\sin.\psi$ represent the cosines of the angles which the plane x'', y'' makes with the plane x, y; z, x; y, z, it follows that where we have the projections c, c', c'' of any area on three coordinate planes, we have its projection $\Sigma m.\left(\frac{x''.dy''-y''.dx''}{dt}\right)$ on the plane of x'', y'', the position of which, with respect to the three planes xy, xz, yz, is given;

it also appears from the expression of $\Sigma m.\left(\frac{x''.dy''-y''.dx''}{dt}\right)$, that for all planes equally inclined to the plane, on which the projection is a maximum, the values of the projection of the area are equal; c, c', c'', being constant, and ∻ to the cosines of the angles, which the plane on which the projection of the area is a maximum, makes with xy, xz, yz, the position of this plane is necessarily fixed and invariable; and as c, c', c'' depend on the coordinates of the bodies at any instant, and on the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, &c. when these quantities are given, we can determine the position of this plane, which may be called invariable because it depends on c, c', c'', which are constant when the bodies are only subject to their mutual action, and to the action of forces directed towards a fixed point. Since the plane x, y, is undetermined in the text, we infer that the sum of the squares of the projections of any areas existing in the invariable plane, on any three coordinate planes existing in the same point of space is constant, therefore if on the axes to the coordinate planes xy, xz, yz, lines be assumed ∻ to c, c', c'', then the diagonal of the parallelopiped whose sides were ∻ to these lines, will represent the quantity and direction of the greatest moment, and this direction is the same whatever three coordinate planes be assumed, but the *position* in absolute space is undetermined, for the projections on all parallel planes are evidently the same. The conclusions to which we have arrived respecting the projections of areas, are evidently applicable to the projections of moments, since, as has been remarked in page 442, these moments may be geometrically reprepresented by triangles, of which the bases represent the projected force, the altitudes being equal to perpendiculars let fall from the point to which the moments are referred, on the directions of the bases; therefore when the forces applied to the several points of the system have an

unique resultant V, since the sum of the moments of any forces projected on a plane is equal to the moment of the projection of their resultant, it follows that the unique resultant V, and the point to which the moments are applied, must exist in the invariable plane, therefore the axis of this plane must be at right angles to this resultant; and as $\frac{P}{V}$, $\frac{Q}{V}$, $\frac{R}{V}$, (see page 443) are equal to the cosines of the angles, which V makes with the coordinates; and as

$$\frac{c}{\sqrt{c^2+c'^2+c''^2}}, \quad \frac{c'}{\sqrt{c^2+c'^2+c''^2}}, \quad \frac{c''}{\sqrt{c^2+c'^2+c''^2}}$$

are equal to the cosines of the angles, which the axis to the invariable plane makes with the same coordinates, we have $\frac{cP+c'Q+c''R}{\sqrt{c^2+c'^2+c''^2}}=0$, (see page 409).

The practical rule for the determination of the plane of greatest projection is given in Chapter II. Vol. II. From what has been established in notes, page 505, it appears that for all points in which $\frac{AdX-BdY}{dt}\cdot\Sigma m=0$, the value of c remains constantly the same; but it is evident that this equation will be satisfied, if the locus of the origin of the coordinates be either the right line described by the centre of gravity or any line parallel to this line; therefore for all such positions the invariable plane remains constantly parallel to itself, however, though for all points of the *same* parallel the direction of the invariable plane remains constantly parallel to itself, still in the passage from one parallel to another, the direction of this plane changes. (A, B, C, are the coordinates of the new origin. *See* Celestial Mechanics, page 145.)

When the forces are reducible to an unique resultant, if the origin of the coordinates be any point in it, the quantities c, c', c'', and therefore the plane, with respect to which the projection of the areas is a maximum, vanishes; and

if the locus of the origin be any line parallel to this line, the value of the projection of the area on the plane x, y, with respect to this line $= \frac{A.dX - B.dY}{dt}. \Sigma m$, for c in this case vanishes, if the locus of the origin be a right line diverging from this resultant, the expression $\frac{A dX - B dY}{dt}.$ Σm, is susceptible of continual increase. The plane, with respect to which the value of $\sqrt{c^2 + c'^2 + c''^2}$ is the minimum maximorum is perpendicular to the direction of the general resultant, or of the common motion with which the system is actuated, its axis is a perpendicular to this plane, erected at the origin, which may be any point in the direction; for all equidistant origins existing in a perpendicular plane, the maximum areas will have the same values, and their planes will be normal to the different generatrices of an hyperboloid of revolution described about this central axis; although the value of the maximum area should be given, still if the origin be not also given, its plane cannot be distinguished from an infinity of others perpendicular to the generatrices of an hyperboloid of revolution; but if with the preceding we combine the condition that the areas should be the *minimum* of the maxima areas, relatively to different origins in space, the plane sought may be easily found, inasmuch as it enjoys not only an exclusive property with respect to those which pass through the same origin, but likewise another exclusive property with respect to those which have the first property common with it.

In the system of the world, as we do not know any fixed point to which the different heavenly bodies may be referred, and as we are also ignorant of the direction and force with which this system moves in space, neither the plane nor the value of the area which is the minimum maximorum can be determined, we can solely select the plane of

the maximum area with respect to *any point* which moves with the velocity of the common centre of gravity of the system in a right line; therefore the origin may be assumed at the common centre of gravity, which, during the entire motion possesses the property of moving in a right line.

The principle of the conservation of areas, and also that of living forces, may be reduced to certain relations between the coordinates of the mutual distances of the bodies composing the system; for if the origin of the coordinates be supposed to be at the centre of gravity, the equation given in page 507 may be made to assume the form

$$c.\Sigma m = \Sigma mm'.\left(\frac{(x'-x).(dy'-dy)-(y-y).dx'-dx)}{dt}\right),$$

$$c'.\Sigma m = \Sigma mm'.\left(\frac{(x'-x).(dz'-dz)-(z'-z).(dx'-dx)}{dt}\right),$$

$$c''.\Sigma m = \Sigma mm'.\left(\frac{(y'-y).(dz'-dz)-(z'-z).(dy'-dy)}{dt}\right),$$

(for the verification of those formula see Celestial Mechanics, page 145), the second members of these equations multiplied by dt, express the sum of the projections of the elementary areas traced by each line which joins the two bodies of the system, of which one is supposed to move round the other considered as immoveable, each area being multiplied by the product of the two masses, which are connected by the same right line. It might be made appear, as in page 508, that the plane passing through any of the bodies of the system, and with respect to which the preceding function is a maximum, remains always parallel to itself, during the motion of the system, and that this plane is parallel to the plane passing through the centre of gravity, relatively to which the function $\Sigma m.\left(\frac{xdy-ydx}{dt}\right)$ is a maximum, &c. Also the second members of the preceding equations vanish with respect to

all planes passing through the same body, and perpendicular to the plane in question.

In like manner the equation given in page 494 may be made to assume the form

$$\Sigma mm'. \left(\frac{(dx'-dx)^2+(dy'-dy)^2+(dz'-dz)^2}{dt^2}\right) =$$
$$C'' - 2\Sigma m.\ \Sigma. \int mm'.\ F.df,$$

which only respects the coordinates of the mutual distances of the bodies, in which the first member expresses the sum of the squares of the relative velocities of the bodies of the system about each other, considering them two by two, and supposing, at the same time, that one of them is immoveable, each square being multiplied by the product of the two masses which are considered. *See* Celestial Mechanics, page 148.

It may be remarked, with respect to the preceding conclusions about the invariable plane, that in any system of solid or fluid molecules actuated primitively by any forces, and subjected to their mutual action, if it happens that after a great number of oscillations these molecules are arranged in a permanent state of rotation about an invariable axis passing through their common centre of gravity, (which is most probably the case with respect to the celestial bodies), then their equator will be parallel to that plane which would furnish the maximum of areas with respect to the centre of gravity. See Vol. II. Chap. IX. page 121.

It may be likewise remarked here, that planes are not the sole surfaces on which the areas remain constant without undergoing any change during the motion of the system; the same property appertains to every circular conic surface, of which the summit is the origin of the radii vectores, but it is necessary that these radii should be projected on the cone by lines parallel to its axis, the areas described on the surfaces of different cones having the same axis and summit, (see page 507), are inversely as the sines of the angles of the

cones, therefore the area will be least, which is projected on the right cone. If the angle of the cone is given but the axes different, there is only one on the surface of which the area traced by the radius vector will be a maximum; also among all those which assign the same value to the maximum areas relatively to different origins in space, there is only one which will give the least of these maxima areas. The axes of these remarkable cones are the same as the axes of the moments or areas which possess the same properties.

END OF THE FIRST VOLUME.

ERRATA.

Page 15, line 12, *from bottom, for* becomes *read* becoming.
—— 21, —— 5, *from bottom, for* and *read* but.
—— 7, —— 3, *for* plan *read* plane.
—— 96, —— 4, *from bottom, after* which *read* is.
—— 114, —— 7, *from bottom, after* from *read* a.
—— 129, —— 13, *for* fuller *read* feebler.
—— 135, —— 4, *dele* of.
—— ib. —— 9, *for* sun *read* earth.
—— 137, —— 8, *after* each *read* other.
—— 150, —— 19, *for* cartonic *read* carbonic.
—— 151, —— 8, *from bottom, for* the contents *read* they.
—— 153, —— 20, *for* transverse *read* traverse.
—— 191, —— 6, *for* the *read* these.
—— 197, —— 9, *for* comets *read* earth.
—— ib. —— 19, *for* on *read* in.
—— 209, —— 5, *after* the *read* periods of the.
—— 210, —— 17, *for* on *read* to.
—— 212, —— 15, *for* first *read* second.
—— 226, —— 14, *after* directions *read* of.
—— 237, —— 17, *for* time *read* velocity.
—— 278, —— 22, *after* all *for* velocity *read* action.